

A new treatment of cross points in Optimized Schwarz Methods

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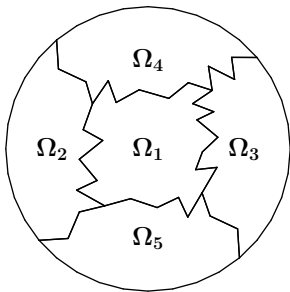
Alpines

Scattering in heterogeneous medium

wave number : $\kappa : \mathbb{R}^d \rightarrow \mathbb{C}$ bounded
 $\Re\{\kappa(\mathbf{x})\} \geq 0, \Im\{\kappa(\mathbf{x})\} \geq 0, \kappa(\mathbf{x}) \neq 0$
source : $f \in L^2(\Omega)$

Non-overlapping partition

$\Omega = \bar{\Omega}_1 \cup \dots \cup \bar{\Omega}_J,$
 $\Gamma_j := \partial\Omega_j, \Gamma'_j := \Gamma_j \setminus \partial\Omega$
 Ω_j : Lipschitz, bounded



Helmholtz bvp

$-\Delta u - \kappa(\mathbf{x})^2 u = f$ in $\Omega,$
 $\partial_n u - \nu \kappa u = 0$ on $\partial\Omega.$



local sub-problems $j = 1 \dots J$

$-\Delta u - \kappa^2 u = f$ in Ω_j
 $\partial_n u - \nu \kappa u = 0$ on $\partial\Omega_j \cap \partial\Omega.$

+

transmission conditions

$\partial_{n_j} u|_{\Gamma_j}^{\text{int}} = -\partial_{n_k} u|_{\Gamma_k}^{\text{int}}$
 $u|_{\Gamma_j}^{\text{int}} = u|_{\Gamma_k}^{\text{int}} \quad \forall j, k$



Cross points allowed

Optimized Schwarz Method (OSM) [Després, 1991]

Optimized Schwarz Method (OSM) is one of the most established DDM approaches for wave propagation. This is a **substructuring method** where transmission conditions are imposed through each interface by means of **Robin traces** involving impedance coefficients.

- **operator valued impedance** : [Collino, Ghanemi & Joly, 2000]
- second order TC : [Gander, Magoules & Nataf, 2002]
- DtN-like impedance : [Nataf, Rogier & de Sturler, 1995], [Antoine, Boudendir & Geuzaine, 2012], [Antoine, Bouajaj & Geuzaine, 2014]
- large literature : overview article [Gander & Zhang, 2019]

Cross point issue

Unappropriate treatment of cross-points may spoil convergence so care must be paid to this issue : [Gander & Kwok, 2013], [Gander & Santugini, 2016], [Després, Nicolopoulos & Thierry, 2020], [Modave, Antoine, Geuzaine & al, 2019 & 2020]. There is also a variant of FETI-DP "à la Després" [Farhat & al, 2005], [Bendali & Boubendir, 2006].

Outline

I Review of the Optimized Schwarz Method

II New manner to enforce transmission condition

III Numerical results

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Original OSM (Després)

Transmission conditions : with scalar $\Lambda > 0$,

$$\begin{aligned} \partial_{n_j} u|_{\Gamma_j} = -\partial_{n_k} u|_{\Gamma_k} & \quad +\partial_{n_j} u|_{\Gamma_j} + \imath\Lambda u|_{\Gamma_j} = \\ u|_{\Gamma_j} = u|_{\Gamma_k} & \quad \iff -\partial_{n_k} u|_{\Gamma_k} + \imath\Lambda u|_{\Gamma_k} \iff \\ \text{on } \Gamma_j \cap \Gamma_k \forall j, k & \quad \text{on } \Gamma_j \cap \Gamma_k \forall j, k \end{aligned} \quad \iff \quad \begin{aligned} (\partial_{n_j} u|_{\Gamma'_j} - \imath\Lambda u|_{\Gamma'_j})_{j=1}^J = \\ -\Pi_0((\partial_{n_k} u|_{\Gamma'_k} + \imath\Lambda u|_{\Gamma'_k})_{k=1}^J) \end{aligned}$$

where the operator Π_0 **swaps traces** on both sides of each interfaces :

$$(v_0, \dots, v_J) = \Pi_0(u_0, \dots, u_J) \iff v_j = u_k \text{ on } \Gamma_j \cap \Gamma_k.$$

Local scattering operators :

$$S_0^{\Gamma_j}(\partial_{n_j} \psi|_{\Gamma'_j} - \imath\Lambda \psi|_{\Gamma'_j}) := \partial_{n_j} \psi|_{\Gamma'_j} + \imath\Lambda \psi|_{\Gamma'_j}$$

for $\Delta \psi + \kappa^2 \psi = 0$ in Ω_j

$$\partial_n \psi - \imath\kappa \psi = 0 \text{ on } \partial\Omega_j \cap \partial\Omega$$

Wave equations :

$$\begin{aligned} (\partial_{n_j} u|_{\Gamma'_j} + \imath\Lambda u|_{\Gamma'_j})_{j=1}^J = \\ S_0((\partial_{n_k} u|_{\Gamma'_k} - \imath\Lambda u|_{\Gamma'_k})_{k=1}^J) + g \end{aligned}$$

$$\text{with } S_0 := \text{diag}_{j=1 \dots J}(S_0^{\Gamma_j}).$$

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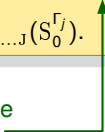
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stems from the source
term of the bvp



Original OSM (Després)

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on $\Gamma_j \cap \Gamma_k \forall j, k$ on $\Gamma_j \cap \Gamma_k \forall j, k$

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Optimized Schwarz

$$(\text{Id} + \Pi_0 S_0) p = -\Pi_0(g)$$

$$\text{with } p = (\partial_{n_j} u|_{\Gamma'_j} - \imath\Lambda u|_{\Gamma'_j})_{j=1}^J$$

The cross-point issue

$$p^{(n+1)} = (1 - r)p^{(n)} - r\Pi_0 S_0(p^{(n)}) - r\Pi_0(g)$$

Without cross points, **geometric convergence** can be obtained for appropriate (operator valued) **impedance Λ** . In practice, convergence is **much slower with cross points** i.e. at best algebraic $\|p - p^{(n)}\|_{L^2} = O(n^{-\gamma})$. The root cause seems related to Π_0 not being continuous at cross-points in proper trace norms.

This so-called "cross point issue" also arises in the different context of multi-domain **boundary integral formulations** where **Multi-Trace Formalism (MTF)** [Claeys & Hiptmair, 2012] now offers a framework that accomodates cross-points, and that is clean as regards function spaces.

Idea : use the **Multi-Trace Formalism** to treat cross-points in OSM. We shall replace Π_0 by a **non-local counterpart Π** that remains continuous no matter the presence of cross points, following the idea first introduced in :



X.Claeys, "Quasi-local multi-trace boundary integral formulations", Numer. Methods Partial Differential Equations, 31(6) :2043–2062, 2015.

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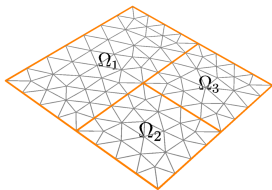
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Discrete function spaces

Triangulation conforming with Ω_j 's, FE spaces $V_h(\Omega_j) = \{ \mathbb{P}_k\text{-Lagrange on } \Omega_j \}$.

Volume functions	Tuples of traces ($\Gamma_j := \partial\Omega_j$)
$V_h(\Omega) := V_h(\Omega_1) \times \cdots \times V_h(\Omega_J)$	$V_h(\Sigma) := V_h(\Gamma_1) \times \cdots \times V_h(\Gamma_J)$
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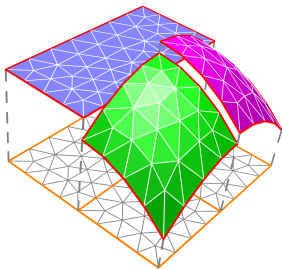


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Piecewise H^1
Possible jumps through interfaces

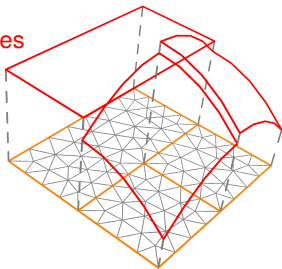


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Multi-traces = tuples of traces at boundaries

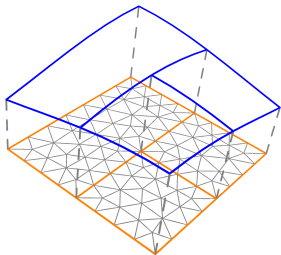


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Single-traces = tuples of traces that match at interfaces



Discrete function spaces

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Impedance

$$t_h(p, q) = t_{\Gamma_1}(p_1, q_1) + \cdots + t_{\Gamma_J}(p_J, q_J)$$

$$t_{\Gamma_j}(\cdot, \cdot) = \text{any coercive sesquilinear form on } V_h(\Gamma_j)$$

Choices of impedance :

- surface mass matrix
- surface order 2 operator
- layer potential, DtN map
- Schur complements
- etc...

For the sake of clarity, $t_h(\cdot, \cdot)$ is assumed SPD.

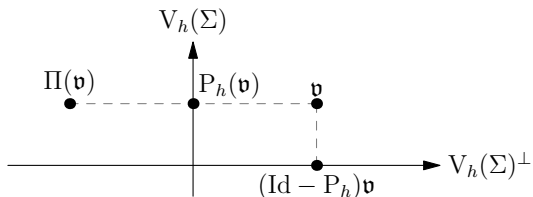
Matching at interfaces via orthogonal symmetry

The t_h -orthogonal projection onto the single-trace space $P_h : \mathbb{V}_h(\Sigma) \rightarrow V_h(\Sigma)$ can be applied by solving a (**DDM friendly!**) SPD problem

$$\begin{aligned} p = P_h(v) \quad \iff \quad & p \in V_h(\Sigma) \quad \text{and} \\ & t_h(p, w) = t_h(v, w) \quad \forall w \in V_h(\Sigma) \end{aligned}$$

Lemma : The t_h -orthogonal symmetry $\Pi := P_h - (\text{Id} - P_h) = 2P_h - \text{Id}$ satisfies $\|\Pi(q)\|_{t_h} = \|q\|_{t_h} \forall q \in \mathbb{V}_h(\Sigma)$ and, for any $v, q \in \mathbb{V}_h(\Sigma)$,

$$(v, q) \in V_h(\Sigma) \times V_h(\Sigma)^\perp \quad \iff \quad q + \iota v = \Pi(-q + \iota v).$$



Reformulations of the scattering problem

Find $u_h \in V_h(\Omega)$ and
 $a(u_h, v_h) = \ell(v_h) \quad \forall v_h \in V_h(\Omega)$



Find $u_h \in \mathbb{V}_h(\Omega)$, $p_h \in \mathbb{V}_h(\Sigma)$ and $\forall v_h \in \mathbb{V}_h(\Omega)$
 $a(u_h, v_h) - \iota t_h(u_h|_\Sigma, v_h|_\Sigma) = t_h(p_h, v_h|_\Sigma) + \ell(v_h)$
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$$a(u, v) := \sum_{j=1}^J \int_{\Omega_j} \nabla u \cdot \nabla \bar{v} - \kappa^2 u \bar{v} dx - \int_{\partial\Omega_j} \iota \kappa u \bar{v} d\sigma$$

$$\ell(v) := \sum_{j=1}^J \int_{\Omega_j} f \bar{v} dx \quad u, v \in \mathbb{V}_h(\Omega)$$

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Non trivial theorem :

- relaxing constraints with Lagrange multipliers
- Use Π to enforce continuity

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Skeleton formulation
 $p_h \in \mathbb{V}_h(\Sigma)$ and
 $(\text{Id} + \Pi S)p_h = g_h$

$$a(u, v) := \sum_{j=1}^J \int_{\Omega_j} \nabla u \cdot \nabla \bar{v} - \kappa^2 u \bar{v} dx - \int_{\partial\Omega_j} \iota \kappa u \bar{v} d\sigma$$

$$\ell(v) := \sum_{j=1}^J \int_{\Omega_j} f \bar{v} dx \quad u, v \in \mathbb{V}_h(\Omega)$$

Unknowns u_h are eliminated in all subdomains in parallel by local "ingoing -to-outgoing" solves, applying a (block diagonal) scattering operator.

Proposition : Define $S(p) := p + 2\iota w|_\Sigma$ where $w \in \mathbb{V}_h(\Omega)$ satisfies $a(w, v) - \iota t_h(w|_\Sigma, v|_\Sigma) = t_h(p, v|_\Sigma) \forall v \in \mathbb{V}_h(\Omega)$. Then $\|S(p)\|_{t_h} \leq \|p\|_{t_h}$ for all $p \in \mathbb{V}_h(\Sigma)$.

Convergence estimate

Theorem :

1) Boundedness : $\|\text{Id} + \Pi S\|_{t_h} \leq 2$

2) Coercivity : $\Re\{t_h(\mathbf{v}, (\text{Id} + \Pi S)\mathbf{v})\} \geq \gamma_h^2 \|\mathbf{v}\|_{t_h}^2 \quad \forall \mathbf{v} \in \mathbb{V}_h(\Sigma)$

Coercivity constant

$$\gamma_h := \frac{\alpha}{\lambda_h^+ + 2 \|\mathbf{a}\| / \lambda_h^-}$$

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$$\alpha := \inf_{u_h, v_h \in \mathbb{V}_h(\Omega) \setminus \{0\}} \sup \frac{|a(u_h, v_h)|}{\|u_h\|_{H^1_{\kappa}(\Omega)} \|v_h\|_{H^1_{\kappa}(\Omega)}}$$

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$$\gamma_h := \frac{\alpha}{\lambda_h^+ + 2 \|a\| / \lambda_h^-}$$

$$\lambda_h^+ := \sup_{\substack{j=1 \dots J \\ v_h \in \mathbb{V}_h(\Gamma_j) \setminus \{0\}}} \frac{t_{\Gamma_j}(v_h, v_h)}{\|v_h\|_{H^1_{\kappa, h}(\Gamma_j)}^2}$$

$$\lambda_h^- := \inf_{\substack{j=1 \dots J \\ v_h \in \mathbb{V}_h(\Gamma_j) \setminus \{0\}}} \frac{t_{\Gamma_j}(v_h, v_h)}{\|v_h\|_{H^1_{\kappa, h}(\Gamma_j)}^2}$$

Convergence estimate

Theorem :

1) Boundedness : $\|\text{Id} + \Pi S\|_{t_h} \leq 2$

2) Coercivity : $\Re\{t_h(\mathbf{v}, (\text{Id} + \Pi S)\mathbf{v})\} \geq \gamma_h^2 \|\mathbf{v}\|_{t_h}^2 \quad \forall \mathbf{v} \in \mathbb{V}_h(\Sigma)$

The exact solution $\mathbf{p}^{(\infty)} \in \mathbb{V}_h(\Sigma)$ to the skeleton formulation can be computed with e.g. a Richardson iteration : given $r \in (0, 1)$, compute

$$\mathbf{p}^{(n+1)} = (1 - r)\mathbf{p}^{(n)} - r\Pi S\mathbf{p}^{(n)} + r\mathbf{g}_h.$$

Proposition : convergence of Richardson's solver

$$\frac{\|\mathbf{p}^{(n)} - \mathbf{p}^{(\infty)}\|_{t_h}}{\|\mathbf{p}^{(0)} - \mathbf{p}^{(\infty)}\|_{t_h}} \leq (1 - 2r(1 - r)\gamma_h^2)^{n/2}.$$

Important consequence : If the $t_{\Gamma_j}(\cdot, \cdot)$'s yield norms that are h -uniformly equivalent to $\|\cdot\|_{\mathbb{H}^{1/2}(\Gamma_j)}$, then we have **h -uniform geometric convergence**.

Only 3 hypothesis required

H1) $\Im m\{a(v, v)\} \leq 0 \quad \forall v \in V_h(\Omega)$ i.e. the medium can only absorb or propagate. The case of "purely" propagative media $\Im m\{a(v, v)\} = 0 \quad \forall v \in V_h(\Omega)$ is covered in particular.

H2) Unique solvability of the discrete pb :

$$\inf_{u_h \in V_h(\Omega)} \sup_{v_h \in V_h(\Omega)} \frac{|a(u_h, v_h)|}{\|u_h\|_{H^1_{\kappa}(\Omega)} \|v_h\|_{H^1_{\kappa}(\Omega)}} > 0.$$

H3) The impedance $t_h(\cdot, \cdot)$ must be chosen coercive.

No assumption on :

- the shape constant of the mesh
- the frequency regime

Outline

I Review of the Optimized Schwarz Method

II New manner to enforce transmission conditions

III Numerical results

Numerical experiments : Helmholtz in 2D

Constant wave number $\kappa > 0$ in a disc $\Omega = D(0, 1)$ and impedance boundary condition $(\partial_n - \nu\kappa)u^{\text{ex}} = g$ with $g(\mathbf{x}) = (\partial_n - \nu\kappa) \exp(-\nu\kappa \mathbf{d} \cdot \mathbf{x})$, discretization with $V_h(\Omega) = \mathbb{P}_1$ -Lagrange.

$$u_h^{\text{ex}} \in V_h(\Omega) \quad \text{and} \quad a(u_h^{\text{ex}}, v_h) = \ell(v_h) \quad \forall v_h \in V_h(\Omega)$$

$$a(u, v) = \int_{\Omega} \nabla u \cdot \nabla \bar{v} - \kappa^2 u \bar{v} d\mathbf{x} - \nu\kappa \int_{\partial\Omega} u \bar{v} d\sigma$$
$$\ell(v) = \int_{\partial\Omega} \bar{v} g d\sigma.$$

With $u_0^{(0)} \equiv 0$, we denote $u_h^{(n)}$ the iterates of the linear solver. The measured error is given by

$$(\text{relative error})^2 = \frac{\sum_{j=1}^J \|u_h^{(n)} - u_h^{\text{ex}}\|_{H^1(\Omega_j)}^2}{\sum_{j=1}^J \|u_h^{(0)} - u_h^{\text{ex}}\|_{H^1(\Omega_j)}^2}.$$

Remarks :

- global linear solver is **GMRes**, relative tolerance = 10^{-8}
- sequential computations on a 6 core workstation
- FEM & DDM code `NIDDL` (in Julia) + `BemTool` (in C++) for integral operators
- exchange operator Π computed with **PCG**.

Choices of impedance

Recall that $t_h(p, q) = t_{\partial\Omega_1}(p_1, q_1) + \dots + t_{\partial\Omega_J}(p_J, q_J)$. We tested several choices of local impedances.

Choice 1 : $M =$ surface mass matrix

$$t_{\partial\Omega_j}(p_h, q_h) = \int_{\partial\Omega_j} p_h(\mathbf{x}) \bar{q}_h(\mathbf{x}) d\sigma(\mathbf{x})$$

This is the impedance originally considered by Després.

Choice 2 : $K =$ surface H1-scalar product

$$t_{\partial\Omega_j}(p_h, q_h) = \int_{\partial\Omega_j} \kappa^{-1} \nabla p_h(\mathbf{x}) \cdot \nabla \bar{q}_h(\mathbf{x}) / 2 + \kappa p_h(\mathbf{x}) \bar{q}_h(\mathbf{x}) d\sigma(\mathbf{x})$$

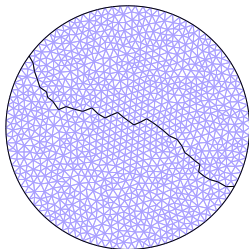
Choice 3 : $W =$ positive hypersingular integral operator

$$t_{\partial\Omega_j}(p_h, q_h) = \int_{\partial\Omega_j \times \partial\Omega_j} \exp(-\kappa|\mathbf{x} - \mathbf{y}|) / (4\pi|\mathbf{x} - \mathbf{y}|) [\kappa^{-1} \mathbf{n}(\mathbf{x}) \times \nabla_{\partial\Omega_j} p_h(\mathbf{x}) \cdot \mathbf{n}(\mathbf{y}) \times \nabla_{\partial\Omega_j} q_h(\mathbf{y}) + \kappa \mathbf{n}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{y}) p_h(\mathbf{x}) q_h(\mathbf{y})] d\sigma(\mathbf{x}, \mathbf{y})$$

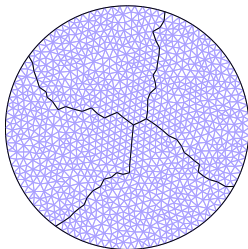
Choice 4 : $\Lambda =$ schur complement (\simeq discrete DtN) associated with the interior numerical solution to the **positive** problem $-\Delta v + \kappa^2 v = 0$ in Ω_j .

Mesh partitioning

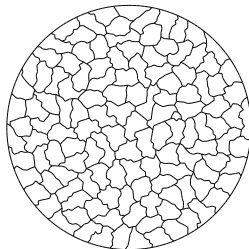
Meshes were generated *a priori* on the whole computational domain with GMSH. Partitioning is obtained *a posteriori* with Metis.



2 subdomains



4 subdomains

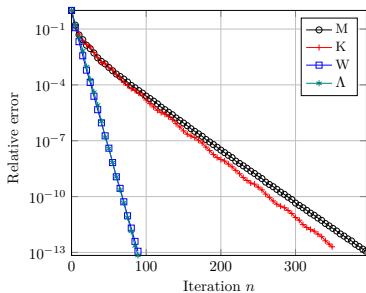


128 subdomains

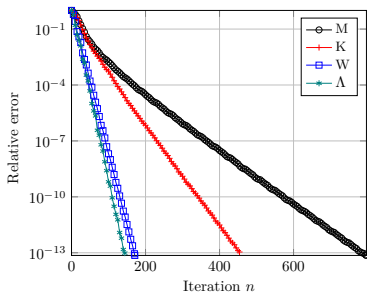
Convergence history

$\kappa = 5$, $\lambda = 2\pi/\kappa \simeq 1.25$
 $N_\lambda = \lambda/h = 40$ points/wavelength.

4 subdomains

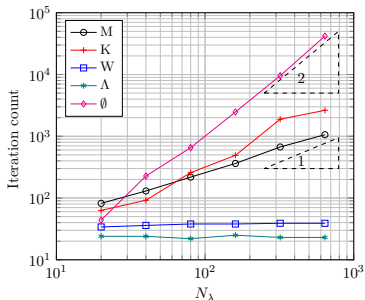


128 subdomains



Iteration count vs. $N_\lambda = \text{points/wavelength}$

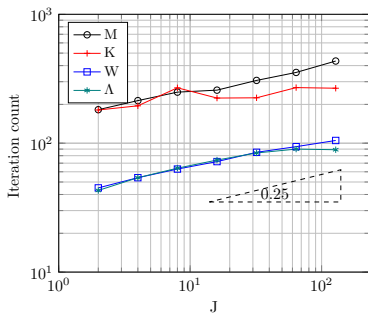
$\kappa = 1$, $\lambda = 2\pi/\kappa \simeq 6.28$, $N_\lambda = \lambda/h$, 4 subdomains.
Relative tolerance of GMRes = 10^{-8} , \emptyset = no DDM.



N_λ	\emptyset	M	K	W	Λ
20	44	82	63	34	24
40	227	130	92	36	24
80	654	218	258	38	22
160	2474	363	491	38	25
320	9559	671	1888	39	23
640	41888	1060	2633	39	23

Iteration count vs. $J =$ number of subdomains

$\kappa = 5$, $\lambda = 2\pi/\kappa \simeq 1.26$, $N_\lambda = \lambda/h = 40$.
Relative tolerance of GMRes = 10^{-8} .

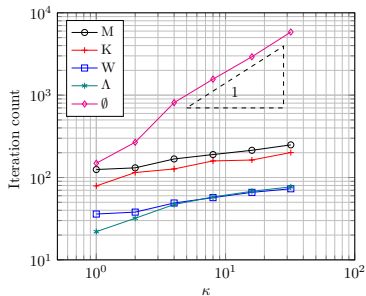


J	M	K	W	A
2	182	181	45	43
4	214	195	54	54
8	249	269	63	64
16	258	224	72	74
32	307	225	85	84
64	354	270	94	90
128	434	267	105	89

Iteration count vs. $\kappa = \text{wavenumber}$

$\lambda = 2\pi/\kappa$, $N_\lambda = \lambda/h = 30$, 4 subdomains.

Relative tolerance of GMRes = 10^{-8} , \emptyset = no DDM.



κ	\emptyset	M	K	W	Λ
1	149	125	79	36	22
2	268	131	115	38	32
4	811	168	127	49	47
8	1563	190	159	57	58
16	2926	214	163	66	68
32	5846	249	201	73	77

Conclusion

We proposed a new way of imposing transmission conditions involving another choice of **exchange operator**. This yields **h -uniform convergence** of iterative solvers, and accomodates **cross-points**.

In addition this approach appears as a **natural generalization** of the classical OSM à la Després, and allows to propose a **full theoretical framework**, which was not available so far.

Also available :

- other boundary conditions (Dirichlet, Neumann),
- other equations (3D Helmholtz, Maxwell),
- analysis of non-infsup-stable impedances.

Future investigations

- fine properties of the exchange operator
- large scale optimized parallel implementation
- multi-level strategy
- non-conforming DDM

Thank you for your attention

Questions ?



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X.Claeys, "Non-self adjoint impedance in Generalized Optimized Schwarz Methods", submitted, preprint Arxiv 2108.03652



X.Claeys, F.Collino and E.Parolin, Matrix form of nonlocal OSM for electromagnetics, preprint Arxiv 2108.11352