Guaranteed error estimates for finite element discretizations of Helmholtz problems

T. Chaumont-Frelet[⋆,†], A. Ern[‡,§], M. Vohralík[§,‡]

June 17,2020

[⋆]Inria Sophia-Antipolis, [†]Laboratoire J.A. Dieudonné, [‡]CERMICS, [§]Inria Paris

Motivations

Motivations

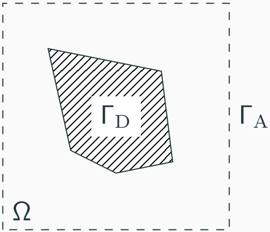
Model problem

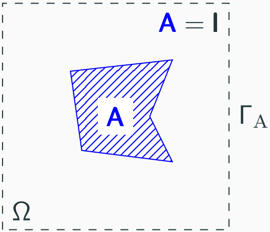
Model problem

Given $f : \Omega \to \mathbb{C}$, find $u : \Omega \to \mathbb{C}$ such that

$$\begin{cases} -\omega^2 \mu u - \nabla \cdot (\mathbf{A} \nabla u) &= \mu f & \text{in } \Omega, \\ u &= 0 & \text{on } \Gamma_{\mathrm{D}}, \\ \mathbf{A} \nabla u \cdot \boldsymbol{n} - i\omega\gamma u &= 0 & \text{on } \Gamma_{\mathrm{A}} \end{cases}$$

where μ, $\mathbf{A}$ and γ are given coefficients strictly positive coefficients.





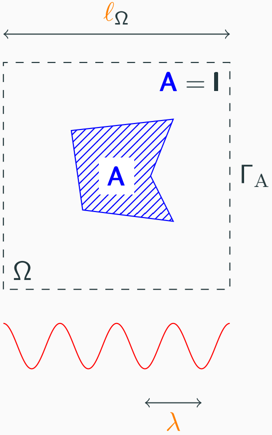
What do I mean by high-frequency?

The physical meaning of μ and $\mathbf{A}$ depends on the application, but the wavespeed is always given by:

$$c := \sqrt{\frac{\sigma_{\min}(\mathbf{A})}{\mu}}$$

The (minimal) wavelength is then given by:

$$\lambda := \frac{2\pi}{\omega} c_{\min}.$$



The "important" quantity is $N_\lambda := \ell_\Omega/\lambda$. High-frequency means that

$$\frac{\omega \ell_\Omega}{c_{\min}} \simeq N_\lambda$$

is "large" (a few tens or hundreds).

Variational formulation

Recall the Helmholtz problem in strong form

$$\begin{cases} -\omega^2 \mu u - \nabla \cdot (\mathbf{A} \nabla u) = \mu f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_{\mathrm{D}}, \\ \mathbf{A} \nabla u \cdot \boldsymbol{n} - i\omega\gamma u = 0 & \text{on } \Gamma_{\mathrm{A}}. \end{cases}$$

Assuming $f \in L^2(\Omega)$, we seek $u \in H^1_{\mathrm{D}}(\Omega)$ such that

$$b(u, v) = (\mu f, v) \quad \forall v \in H^1_{\mathrm{D}}(\Omega),$$

with

$$b(u, v) := -\omega^2 (\mu u, v)_\Omega - i\omega(\gamma u, v)_{\Gamma_{\mathrm{A}}} + (\mathbf{A} \nabla u, \nabla v)_\Omega.$$

Finite element approximation

We consider a mesh $\mathcal{T}_h$ of Ω into tetrahedral element K.

The elements $K \in \mathcal{T}_h$ are "small" ($h_K := \operatorname{diam} K \leq h$).

The coefficients $\mu, \gamma, \mathbf{A}$ are constant inside each element/face.

We introduce a "finite element" discretization space

$$V_h := \left\{ v_h \in H^1_{\mathrm{D}}(\Omega) \mid v_h|_K \in \mathbb{P}_p(K) \; \forall K \in \mathcal{T}_h \right\}$$

with $p \geq 1$.

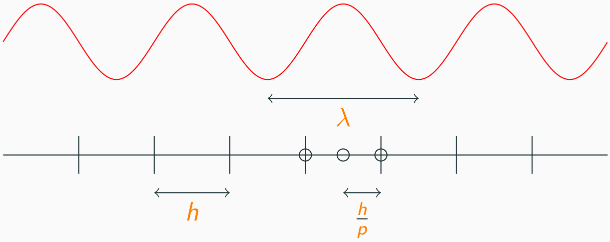
P.G. Ciarlet, 2002.

What do I mean by refined mesh?

When I am saying that "the mesh is fine" , I mean that

$$N_{\mathrm{dofs}/\lambda} = \lambda \Big/ \frac{h}{p} \simeq \left(\frac{\omega h}{c_{\min} p}\right)^{-1}$$

is large.



Finite element approximation

Recall that u is the only element of $H^1_{\mathrm{D}}(\Omega)$ such that

$$b(u, v) = (\mu f, v) \quad \forall v \in H^1_{\mathrm{D}}(\Omega).$$

Analogously, we seek a discrete a discrete solution $u_h \in V_h$ such that

$$b(u_h, v_h) = (\mu f, v_h) \quad \forall v_h \in V_h. \tag{1}$$

Problem (1) corresponds to a finite dimensional linear system, that we can numerically solve.

In this talk, we are especially interested in measuring the error

$$e_h := u - u_h$$

Motivations

A priori error estimates

A priori error estimates

A priori estimates

Assume that $\omega h / c_{\min} p \leq \mathscr{C}_1$, then

$$\|\nabla e_h\|_{\mathbf{A},\Omega} \leq \mathscr{C}_2 \left(\frac{\omega h}{c_{\min} p}\right)^p.$$

F. Ihlenburg and I. Babuška, *SIAM J. Numer. Anal.*, 1997.

J.M. Melenk and S.A. Sauter, *Math. Comp.*, 2010.

T. Chaumont-Frelet and S. Nicaise, *IMA J. Numer. Anal.*, 2019.

Some limitations:

- The above result requires important regularity assumptions.
- The error estimate is not always applicable.
- The constants $\mathscr{C}_1$ and $\mathscr{C}_2$ are *not* computable in general.

A priori error estimates

A priori estimates provide qualitative upper bounds.

They are important as they show that the method converges.
They also indicate how fast the convergence happens.

They are not suited to *quantitatively* estimate the error in practice.

Motivations

A posteriori error estimation

A posteriori estimates

(ideal) A posteriori estimates

$$\|\boldsymbol{\nabla} e_h\|_{\mathbf{A},\Omega} \leq \eta$$

Here η is a fully-computable real number called an "error estimator".

This quantity is computed as a post-processing of u_h, i.e. $\eta = \eta(u_h)$.

There is *no generic constants*. We have a *guaranteed* error estimate.

Outline

1. Low frequencies
2. High frequencies
3. Controlling the pre-factors
4. Numerical illustrations

Low frequencies

The low frequency case

We first consider the low frequency limit where $\omega = 0$.

The problem then reads: find $u : \Omega \to \mathbb{C}$ such that

$$\begin{cases} -\boldsymbol{\nabla} \cdot (\mathbf{A} \boldsymbol{\nabla} u) &= \mu f & \text{in } \Omega, \\ u &= 0 & \text{on } \Gamma_{\mathrm{D}}, \\ \mathbf{A} \boldsymbol{\nabla} u \cdot \boldsymbol{n} &= 0 & \text{on } \Gamma_{\mathrm{A}}. \end{cases}$$

For the sake of simplicity, we will assume that $f = f_h \in \mathbb{P}_p(\mathcal{T}_h)$.

Variational formulation and discrete solution

Consider the sesquilinear form

$$a(u, v) = (\mathbf{A}\boldsymbol{\nabla} u, \boldsymbol{\nabla} v)_\Omega, \quad u, v \in H^1_{\Gamma_{\mathrm{D}}}(\Omega).$$

We can characterize u as the unique element of $H^1_{\Gamma_{\mathrm{D}}}(\Omega)$ such that

$$a(u, v) = (\mu f_h, v)_\Omega$$

for all $v \in H^1_{\Gamma_{\mathrm{D}}}(\Omega)$.

u_h is the unique element of V_h satisfying

$$a(u_h, v_h) = (\mu f_h, v_h)_\Omega$$

for all $v_h \in V_h$.

Low frequencies

Key ideas behind a posteriori estimation

Key ideas

Second-order problems typically arise from two physical laws,
e.g., "Faraday's law + Gauss' law = Poisson problem".

The continuous solution u is uniquely determined by the condition that

$$u \in H^1(\Omega), \qquad u = 0 \text{ on } \Gamma_{\mathrm{D}} \tag{2a}$$

and for the "flux" $\boldsymbol{\sigma} := -\mathbf{A}\nabla u \in \boldsymbol{H}(\mathrm{div}, \Omega)$

$$\boldsymbol{\sigma} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma_{\mathrm{A}}, \qquad \nabla \cdot \boldsymbol{\sigma} = \mu f_h \text{ in } \Omega. \tag{2b}$$

The discrete solution u_h satisfies (2a) but not (2b) in general.

Key ideas

Consider the minimization problem

$$\boldsymbol{\sigma} := \arg \min_{\substack{\boldsymbol{\tau}\in \boldsymbol{H}(\mathrm{div},\Omega) \\ \boldsymbol{\tau}\cdot\boldsymbol{n}=0 \text{ on } \Gamma_{\mathrm{A}} \\ \nabla\cdot\boldsymbol{\tau}=\mu f_h \text{ in } \Omega}} \|\mathbf{A}^{-1}\boldsymbol{\tau} + \nabla u_h\|_{\mathbf{A},\Omega}.$$

If the above minimum is 0, then $\boldsymbol{\sigma} = -\mathbf{A}\nabla u_h \in \boldsymbol{H}(\mathrm{div},\Omega)$ with

$$\boldsymbol{\sigma}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma_{\mathrm{A}} \qquad \nabla\cdot\boldsymbol{\sigma} = \mu f_h \text{ in } \Omega$$

i.e. u_h satisfies (2b), which implies that $u = u_h$.

Otherwise, it measures how “non-conforming” u_h is.

Low frequencies

The Prager-Synge theorem

Prager-Synge inequality

Assume that $\boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div}, \Omega)$ satisfies

$$\boldsymbol{\sigma} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma_{\mathrm{A}}, \qquad \nabla \cdot \boldsymbol{\sigma} = \mu f_h \text{ in } \Omega.$$

$\boldsymbol{\sigma} = -\mathbf{A}\nabla u$ is *one* example.

Then

$$\begin{aligned} a(e_h, v) &= (\mu f_h, v)_\Omega - (\mathbf{A}\nabla u_h, \nabla v)_\Omega \\ &= (\nabla \cdot \boldsymbol{\sigma}, v)_\Omega - (\mathbf{A}\nabla u_h, \nabla v)_\Omega \\ &= -(\boldsymbol{\sigma} + \mathbf{A}\nabla u_h, \nabla v)_\Omega. \end{aligned}$$

Prager–Synge inequality

$$|a(e_h, v)| \leq \|\mathbf{A}^{-1}\boldsymbol{\sigma} + \nabla u_h\|_{\mathbf{A},\Omega} \|\nabla v\|_{\mathbf{A},\Omega}$$

Upper bound via equilibrated flux

Recall that whenever $\nabla \cdot \boldsymbol{\sigma} = \mu f_h$ in Ω and $\boldsymbol{\sigma} \cdot \boldsymbol{n} = 0$ on Γ_{A}

$$|a(e_h, v)| \leq \|\mathbf{A}^{-1}\boldsymbol{\sigma} + \nabla u_h\|_{\mathbf{A},\Omega}\|\nabla v\|_{\mathbf{A},\Omega} \quad \forall v \in H^1_{\Gamma_{\mathrm{D}}}(\Omega).$$

Picking in particular $v = e_h$, we have

$$\|\nabla e_h\|^2_{\mathbf{A},\Omega} = |a(e_h, e_h)| \leq \|\mathbf{A}^{-1}\boldsymbol{\sigma} + \nabla u_h\|_{\mathbf{A},\Omega}\|\nabla e_h\|_{\mathbf{A},\Omega}.$$

Guaranteed upper bound

$$\|\nabla e_h\|_{\mathbf{A},\Omega} \leq \|\mathbf{A}^{-1}\boldsymbol{\sigma} + \nabla u_h\|_{\mathbf{A},\Omega}$$

W. Prager and J.L. Synge, *Quart. Appl. Math.*, 1947.

Low frequencies

Practical flux construction

Practical flux construction

Equilibrated fluxes satisfy $\nabla \cdot \boldsymbol{\sigma} = \mu f_h$ in Ω and $\boldsymbol{\sigma} \cdot \boldsymbol{n} = 0$ on Γ_{A}.
They readily provide guaranteed error bounds.

Here, because $\mu f_h \in \mathbb{P}_p(\mathcal{T}_h)$ we can actually find discrete fluxes $\boldsymbol{\sigma}_h$.

The correct tool to do that is the Raviart–Thomas finite element space

$$\boldsymbol{W}_h := \left\{ \boldsymbol{w}_h \in \boldsymbol{H}_{\Gamma_{\mathrm{A}}}(\mathrm{div}, \Omega) \mid \boldsymbol{w}_h|_K \in [\mathbb{P}_p(K)]^3 + \boldsymbol{x}\mathbb{P}_{p-1}(K) \right\}.$$

P.A. Raviart and J.M. Thomas, 1977.

How to select the flux?

We are going to select a discrete flux $\boldsymbol{\sigma}_h \in \boldsymbol{W}_h$.

The "equilibration" constraint on the flux is

$$\nabla \cdot \boldsymbol{\sigma}_h = \mu f_h \text{ in } \Omega.$$

The estimate the flux provides us is

$$\|\nabla e_h\|_{\mathbf{A},\Omega} \leq \|\mathbf{A}^{-1}\boldsymbol{\sigma}_h + \nabla u_h\|_{\mathbf{A},\Omega}.$$

Optimal discrete flux

Recall that $\boldsymbol{\sigma}_h \in \boldsymbol{W}_h$ has to satisfy

$$\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}_h = \mu f_h \text{ in } \Omega$$

and gives us

$$\|\boldsymbol{\nabla} e_h\|_{\mathbf{A},\Omega} \leq \|\mathbf{A}^{-1}\boldsymbol{\sigma}_h + \boldsymbol{\nabla} u_h\|_{\mathbf{A},\Omega}.$$

The "optimal" choice is then

$$\boldsymbol{\sigma}_h := \arg \min_{\substack{\boldsymbol{\tau}_h \in \boldsymbol{W}_h \\ \boldsymbol{\nabla}\cdot\boldsymbol{\tau}_h = \mu f_h \text{ in } \Omega}} \|\mathbf{A}^{-1}\boldsymbol{\tau}_h + \boldsymbol{\nabla} u_h\|_{\mathbf{A},\Omega}.$$

How to compute it and how expensive it is?

After introducing a Lagrange multiplier, there exists a unique pair $(\boldsymbol{\sigma}_h, q_h) \in \boldsymbol{W}_h \times \mathbb{P}_p(\mathcal{T}_h)$ such that

$$\left\{ \begin{array}{rcll} (\mathbf{A}^{-1}\boldsymbol{\sigma}_h, \boldsymbol{w}_h)_\Omega + (q_h, \nabla \cdot \boldsymbol{w}_h)_\Omega & = & -(\nabla u_h, \boldsymbol{w}_h)_\Omega & \forall \boldsymbol{w}_h \in \boldsymbol{W}_h, \\ (\nabla \cdot \boldsymbol{\sigma}_h, r_h) & = & (\mu f_h, r_h) & \forall r_h \in \mathbb{P}_p(\mathcal{T}_h). \end{array} \right.$$

This square linear system can be solved to compute the optimal flux.

Unfortunately, it is more expensive than solving the original problem, so we avoid that in practice by using a localization trick.

P. Destuynder and B. Métivet, *Math. Comp.*, 1999.

Localization

At the continuous level, the ideal flux is $\boldsymbol{\sigma} := -\mathbf{A}\boldsymbol{\nabla} u$.

A characterization is

$$\boldsymbol{\sigma} = \arg \min_{\substack{\boldsymbol{\tau} \in \boldsymbol{H}_{\Gamma_{\mathrm{A}}}(\mathrm{div},\Omega) \\ \boldsymbol{\nabla}\cdot\boldsymbol{\tau} = \mu f_h \text{ in } \Omega}} \|\mathbf{A}^{-1}\boldsymbol{\tau} + \boldsymbol{\nabla} u\|_{\mathbf{A},\Omega}.$$

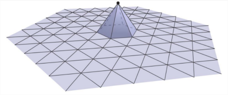
The "optimal" flux directly mimicks this definition at the discrete level

$$\boldsymbol{\sigma}_h := \arg \min_{\substack{\boldsymbol{\tau}_h \in \boldsymbol{W}_h \\ \boldsymbol{\nabla}\cdot\boldsymbol{\tau}_h = \mu f_h \text{ in } \Omega}} \|\mathbf{A}^{-1}\boldsymbol{\tau}_h + \boldsymbol{\nabla} u_h\|_{\mathbf{A},\Omega}.$$

Localization

Consider the set of "hat functions" $\{\psi^{\boldsymbol{a}}\}_{\boldsymbol{a}\in\mathcal{V}_h}$ of the mesh. We then have

$$\sum_{\boldsymbol{a}\in\mathcal{V}_h} \psi^{\boldsymbol{a}} = 1.$$



The ideal flux $\boldsymbol{\sigma} := -\mathbf{A}\boldsymbol{\nabla} u$ can be decomposed as

$$\boldsymbol{\sigma} = \sum_{\boldsymbol{a}\in\mathcal{V}_h} \boldsymbol{\sigma}^{\boldsymbol{a}}, \qquad \boldsymbol{\sigma}^{\boldsymbol{a}} = -\psi^{\boldsymbol{a}}\mathbf{A}\boldsymbol{\nabla} u.$$

Easy computations show that

$$\boldsymbol{\sigma}^{\boldsymbol{a}} \cdot \boldsymbol{n} = 0 \text{ on } \partial\omega^{\boldsymbol{a}} \qquad \boldsymbol{\nabla} \cdot \boldsymbol{\sigma}^{\boldsymbol{a}} = \psi^{\boldsymbol{a}}\mu f_h - \mathbf{A}\boldsymbol{\nabla} u \cdot \boldsymbol{\nabla}\psi^{\boldsymbol{a}} \text{ in } \omega^{\boldsymbol{a}}$$

Localization

We have shown that

$$\boldsymbol{\sigma} = \sum_{\boldsymbol{a} \in \mathcal{V}_h} \boldsymbol{\sigma}^{\boldsymbol{a}}$$

and

$$\boldsymbol{\sigma}^{\boldsymbol{a}} = \arg \min_{\substack{\boldsymbol{\tau} \in \boldsymbol{H}_0(\mathrm{div}, \omega^{\boldsymbol{a}}) \\ \nabla \cdot \boldsymbol{\tau} = \psi^{\boldsymbol{a}} \mu f_h - \mathbf{A} \nabla u \cdot \nabla \psi^{\boldsymbol{a}} \text{ in } \omega^{\boldsymbol{a}}}} \|\mathbf{A}^{-1} \boldsymbol{\tau} + \nabla u\|_{\mathbf{A}, \omega^{\boldsymbol{a}}}.$$

We can mimick that on the discrete level!

Localization

We thus set

$$\boldsymbol{\sigma}_h := \sum_{\boldsymbol{a} \in \mathcal{V}_h} \boldsymbol{\sigma}_h^{\boldsymbol{a}}$$

with

$$\boldsymbol{\sigma}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{\tau}_h \in \boldsymbol{H}_0(\mathrm{div}, \omega^{\boldsymbol{a}}) \cap \boldsymbol{W}_h \\ \boldsymbol{\nabla} \cdot \boldsymbol{\tau}_h = \psi^{\boldsymbol{a}} \mu f_h - \mathbf{A} \boldsymbol{\nabla} u_h \cdot \boldsymbol{\nabla} \psi^{\boldsymbol{a}} \text{ in } \omega^{\boldsymbol{a}}}} \|\mathbf{A}^{-1} \boldsymbol{\tau}_h + \boldsymbol{\nabla} u_h\|_{\mathbf{A}, \omega^{\boldsymbol{a}}}.$$

The compatibility condition

$$(\psi^{\boldsymbol{a}} \mu f_h - \mathbf{A} \boldsymbol{\nabla} u_h \cdot \boldsymbol{\nabla} \psi^{\boldsymbol{a}}, 1)_{\omega^{\boldsymbol{a}}} = (\mu f_h, \psi^{\boldsymbol{a}})_\Omega - (\mathbf{A} \boldsymbol{\nabla} u_h, \boldsymbol{\nabla} \psi^{\boldsymbol{a}})_\Omega = 0$$

holds true since u_h is the discrete solution and $\psi^{\boldsymbol{a}} \in V_h$.

Summary of the localization process

Step 1: solve a set of small, uncoupled linear systems

$$\boldsymbol{\sigma}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{\tau}_h \in \boldsymbol{H}_0(\mathrm{div}, \omega^{\boldsymbol{a}}) \cap \boldsymbol{W}_h \\ \boldsymbol{\nabla} \cdot \boldsymbol{\tau}_h = \psi^{\boldsymbol{a}} \mu f_h - \mathbf{A} \boldsymbol{\nabla} u_h \cdot \boldsymbol{\nabla} \psi^{\boldsymbol{a}} \text{ in } \omega^{\boldsymbol{a}}}} \|\mathbf{A}^{-1} \boldsymbol{\tau}_h + \boldsymbol{\nabla} u_h\|_{\mathbf{A}, \omega^{\boldsymbol{a}}}.$$

Step 2: assemble these local contributions

$$\boldsymbol{\sigma}_h := \sum_{\boldsymbol{a} \in \mathcal{V}_h} \boldsymbol{\sigma}_h^{\boldsymbol{a}}.$$

Step 3: compute the estimator

$$\eta := \|\mathbf{A}^{-1} \boldsymbol{\sigma}_h + \boldsymbol{\nabla} u_h\|_{\mathbf{A}, \Omega}.$$

Step 4: enjoy the guaranteed estimate

$$\|\boldsymbol{\nabla} e_h\|_{\mathbf{A}, \Omega} \leq \eta.$$

Low frequencies

Efficiency

Efficiency

For time reasons, I will not give any proofs, but we can show that

$$\eta \leq C_{\text{eff}} \|\nabla e_h\|_{\mathbf{A},\Omega},$$

where C_{eff} only depends on:

- the "flatness" of the tetrahedra in the mesh,
- the "contrasts" in the coefficients.

A nice aspect is that C_{eff} does not depend on p.

P. Braess, V. Pillwein and J. Schöberl, *CMAME*, 2009.

A. Ern and M. Vohralìk, *Math. Comp.*, 2020.

Low frequencies

Takeaways

Takeaways

An equilibrated flux is an object $\boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div}, \Omega)$ such that

$$\boldsymbol{\sigma} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma_{\mathrm{A}} \qquad \boldsymbol{\nabla} \cdot \boldsymbol{\sigma} = \mu f_h \text{ in } \Omega.$$

There exist efficient algorithms to build a discrete flux $\boldsymbol{\sigma}_h \in \boldsymbol{W}_h$.

The *guaranteed* error estimate

$$\|\boldsymbol{\nabla} e_h\|_{\mathbf{A},\Omega} \leq \eta$$

holds true with

$$\eta := \|\mathbf{A}^{-1}\boldsymbol{\sigma}_h + \boldsymbol{\nabla} u_h\|_{\mathbf{A},\Omega}.$$

This bound cannot be too loose:

$$\eta \leq C_{\mathrm{eff}}\|\boldsymbol{\nabla} e_h\|_{\mathbf{A},\Omega}.$$

High frequencies

The high frequency case

Back to our original problem

$$\begin{cases} -\omega^2 \mu u - \boldsymbol{\nabla} \cdot (\mathbf{A} \boldsymbol{\nabla} u) & = & \mu f & \text{in } \Omega, \\ u & = & 0 & \text{on } \Gamma_{\mathrm{D}}, \\ \mathbf{A} \boldsymbol{\nabla} u \cdot \boldsymbol{n} - i\omega\gamma u & = & 0 & \text{on } \Gamma_{\mathrm{A}}. \end{cases}$$

We introduce

$$b(u, v) := -\omega^2 (\mu u, v)_\Omega - i\omega(\gamma u, v)_{\Gamma_{\mathrm{A}}} + (\mathbf{A} \boldsymbol{\nabla} u, \boldsymbol{\nabla} v)_\Omega.$$

Energy norm and lack of coercivity

We will consider the "balanced" norm

$$|||v|||^2_{\omega,\Omega} := \omega^2 \|v\|^2_{\mu,\Omega} + \|\nabla v\|^2_{\mathbf{A},\Omega}.$$

The sesquilinear form b is not coercive.

Instead we have the "Gårding" inequality

$$\operatorname{Re} b(v,v) = \|\nabla v\|^2_{\mathbf{A},\Omega} - \omega^2 \|v\|^2_{\mu,\Omega} = |||v|||^2_{\omega,\Omega} - 2\omega^2 \|v\|^2_{\mu,\Omega}.$$

High frequencies

Flux equilibration

Definition of an equilibrated flux

Letting $\boldsymbol{\sigma} := -\mathbf{A}\nabla u$, we have

$$\boldsymbol{\sigma} \cdot \boldsymbol{n} = -i\omega\gamma u \text{ on } \Gamma_{\mathrm{A}} \qquad \nabla \cdot \boldsymbol{\sigma} = \mu f_h + \omega^2 \mu u \text{ in } \Omega.$$

Hence, natural requirements for $\boldsymbol{\sigma}_h$ are

$$\boldsymbol{\sigma}_h \cdot \boldsymbol{n} = -i\omega\gamma u_h \text{ on } \Gamma_{\mathrm{A}} \qquad \nabla \cdot \boldsymbol{\sigma}_h = \mu f_h + \omega^2 \mu u_h \text{ in } \Omega.$$

The “standard” reconstruction algorithms directly extend.

Prager-Synge inequality

Let $v \in H^1_{\Gamma_{\mathrm{D}}}(\Omega)$. We have

$$\begin{aligned} b(e_h, v) &= (\mu f_h, v)_\Omega - b(u_h, v) \\ &= (\mu f_h + \omega^2 \mu u_h, v)_\Omega + i\omega(\gamma u_h, v)_{\Gamma_{\mathrm{A}}} - (\mathbf{A}\boldsymbol{\nabla} u_h, \boldsymbol{\nabla} v)_\Omega \\ &= (\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}_h, v)_\Omega - (\boldsymbol{\sigma}_h \cdot \boldsymbol{n}, v)_{\Gamma_{\mathrm{A}}} - (\mathbf{A}\boldsymbol{\nabla} u_h, \boldsymbol{\nabla} v)_\Omega \\ &= -(\boldsymbol{\sigma}_h + \mathbf{A}\boldsymbol{\nabla} u_h, \boldsymbol{\nabla} v)_\Omega. \end{aligned}$$

Prager-Synge inequality

$$|b(e_h, v)| \le \eta \|\boldsymbol{\nabla} v\|_{\mathbf{A},\Omega} \quad \forall v \in H^1_{\Gamma_{\mathrm{D}}}(\Omega)$$

So far, so good!

What's the matter?

Here, we do not have

$$\|\boldsymbol{\nabla} e_h\|^2_{\mathbf{A},\Omega} \leq |b(e_h, e_h)|,$$

which is a major issue!

Instead, we only have the "Gårding" inequality

$$\operatorname{Re} b(e_h, e_h) \geq |||e_h|||^2_{\omega,\Omega} - 2\omega^2 \|e_h\|^2_{\mu,\Omega}.$$

High frequencies

A coarse error estimate

Stability constant

For $g \in L^2(\Omega)$, let $\mathscr{S}^\star g$ denote the unique element of $H^1_{\Gamma_{\mathrm{D}}}(\Omega)$ such that

$$b(w, \mathscr{S}^\star g) = 2\omega^2(\mu w, g)_\Omega \quad \forall w \in H^1_{\Gamma_{\mathrm{D}}}(\Omega)$$

and let

$$\mathscr{C}_{\mathrm{st}} := \frac{1}{\omega} \max_{\substack{g \in L^2(\Omega) \\ \|g\|_{\mu,\Omega}=1}} \|\boldsymbol{\nabla}(\mathscr{S}^\star g)\|_{\mathbf{A},\Omega}.$$

$\mathscr{C}_{\mathrm{st}}$ is the best constant such that

$$\|\boldsymbol{\nabla}(\mathscr{S}^\star g)\|_{\mathbf{A},\Omega} \leq \mathscr{C}_{\mathrm{st}}\omega\|g\|_{\mu,\Omega} \quad \forall g \in L^2(\Omega).$$

It is closely related to resolvant estimates.

Making up for the lack of coercivity

By definition, we have

$$b(w, \mathscr{S}^\star e_h) = 2\omega^2(\mu w, e_h) \quad \forall w \in H^1_{\Gamma_{\mathrm{D}}}(\Omega).$$

Hence, in particular,

$$b(e_h, \mathscr{S}^\star e_h) = 2\omega^2 \|e_h\|^2_{\mu,\Omega},$$

which is exactly the "bad" term the Gårding inequality:

$$\operatorname{Re} b(e_h, e_h) = |||e_h|||^2_{\omega,\Omega} - 2\omega^2 \|e_h\|^2_{\mu,\Omega}.$$

Making up for the lack of coercivity

Using Prager-Synge inequality, we have

$$\|\!|e_h|\!\|^2_{\omega,\Omega} = \operatorname{Re} b(e_h, e_h + \mathscr{S}^\star e_h) \leq \eta \|\boldsymbol{\nabla}(e_h + \mathscr{S}^\star e_h)\|_{\mathbf{A},\Omega}.$$

It follows that

$$\begin{aligned} \|\!|e_h|\!\|^2_{\omega,\Omega} &\leq \eta \left(\|\boldsymbol{\nabla} e_h\|_{\mathbf{A},\Omega} + \|\boldsymbol{\nabla}(\mathscr{S}^\star e_h)\|_{\mathbf{A},\Omega}\right) \\ &\leq \eta \left(\|\boldsymbol{\nabla} e_h\|_{\mathbf{A},\Omega} + \mathscr{C}_{\mathrm{st}}\omega\|e_h\|_{\mu,\Omega}\right) \\ &\leq \eta \max(1, \mathscr{C}_{\mathrm{st}}) \|\!|e_h|\!\|_{\omega,\Omega}, \end{aligned}$$

and

$$\|\!|e_h|\!\|_{\omega,\Omega} \leq \max(1, \mathscr{C}_{\mathrm{st}})\eta.$$

Coarse error estimate

We obtained the following error estimate:

Coarse error estimate

$$\|e_h\| \leq \max(1, \mathscr{C}_{\mathrm{st}})\eta$$

$\mathscr{C}_{\mathrm{st}}$ is the best constant such that:

Stability constant

$$\|\boldsymbol{\nabla}(\mathscr{S}^{\star} g)\|_{\mathbf{A},\Omega} \leq \mathscr{C}_{\mathrm{st}}\omega\|g\|_{\mu,\Omega} \qquad \forall g \in L^2(\Omega).$$

High frequencies

Efficiency

Efficiency

We can show that

$$\eta \leq C_{\mathrm{eff}} \left(1 + \max_{K \in \mathcal{T}_h} \frac{\omega h_K}{c_{\min K}\, p} \right) |||e_h|||_{\omega,\Omega} \,,$$

where $c_{\min K}$ is the wavespeed in the element K.

For any reasonable discretization, we have

$$\frac{\omega h_K}{c_{\min K}\, p} \leq 1,$$

so that in practice

$$\eta \leq C_{\mathrm{eff}} \, |||e_h|||_{\omega,\Omega} \,,$$

where C_{eff} only depends on the elements "flatness" and the contrasts.

T. Chaumont-Frelet, A. Ern and M. Vohralík, *Numer. Math.*, 2021.

W. Dörfler, S. Sauter, *Comput. Meth. Appl. Math.*, 2013.

The problem with the coarse error estimate

Recall that

$$\eta \leq C_{\text{eff}} \, |||e_h|||_{\omega,\Omega} \qquad |||e_h|||_{\omega,\Omega} \leq \max(1, \mathscr{C}_{\text{st}})\eta.$$

We have $C_{\text{eff}} \simeq 1$ and $\mathscr{C}_{\text{st}} \gtrsim \omega\ell_\Omega / c_{\min}$, so that

$$\eta \lesssim |||e_h|||_{\omega,\Omega} \lesssim \frac{\omega\ell_\Omega}{c_{\min}}\eta.$$

In practice, the coarse error estimate will largely overestimate the error in the high frequency regime.

High frequencies

Sharp error estimate

The approximation factor

We introduce

$$\mathscr{C}_{\mathrm{ap}} := \frac{1}{\omega} \max_{\substack{g \in L^2(\Omega) \\ \|g\|_{\mu,\Omega}=1}} \min_{v_h \in V_h} \|\boldsymbol{\nabla}(\mathscr{S}^\star g - v_h)\|_{\mathbf{A},\Omega}.$$

Approximability

For all $g \in L^2(\Omega)$, there exists $v_h^\star \in V_h$ such that

$$\|\boldsymbol{\nabla}(\mathscr{S}^\star g - v_h^\star)\|_{\mathbf{A},\Omega} \leq \mathscr{C}_{\mathrm{ap}} \omega \|g\|_{\mu,\Omega}.$$

This was called η in Markus' talk.

It's better than the stability constant!

Recall that

$$\mathscr{C}_{\rm ap} := \max_{\substack{g \in L^2(\Omega) \\ \|g\|_{\mu,\Omega}=1}} \min_{v_h \in V_h} \omega \, |||\mathscr{S} g - v_h|||_{\omega,\Omega}$$

since we can take $v_h = 0$, we have

$$\mathscr{C}_{\rm ap} \leq \max_{\substack{g \in L^2(\Omega) \\ \|g\|_{\mu,\Omega}=1}} \omega \, |||\mathscr{S} g|||_{\omega,\Omega} =: \mathscr{C}_{\rm st}.$$

Besides, standard approximability results for FEM show that

$$\mathscr{C}_{\rm ap} \leq \mathscr{C} \left(\frac{1}{p} \frac{h}{\ell_\Omega} \right)^s$$

for some $s > 0$, so that $\mathscr{C}_{\rm ap} \to 0$.

Using Galerkin orthogonality

Recall that

$$|||e_h|||_{\omega,\Omega}^2 = \operatorname{Re} b(e_h, e_h + \mathscr{S}^\star e_h).$$

By Galerkin orthogonality, we have

$$\begin{aligned}
|||e_h|||_{\omega,\Omega}^2 &= \operatorname{Re} b(e_h, e_h) + \operatorname{Re} b(e_h, \mathscr{S}^\star e_h) \\
&= \operatorname{Re} b(e_h, e_h) + \operatorname{Re} b(e_h, \mathscr{S}^\star e_h - v_h^\star) \\
&\leq \eta \left(\|\boldsymbol{\nabla} e_h\|_{\mathbf{A},\Omega} + \|\boldsymbol{\nabla}(\mathscr{S}^\star e_h - v_h^\star)\|_{\mathbf{A},\Omega} \right) \\
&\leq \eta \max(1, \mathscr{C}_{\mathrm{ap}}) \, |||e_h|||_{\omega,\Omega} \,.
\end{aligned}$$

Sharp error estimate

Sharp error estimate

$$|\!|\!| e_h |\!|\!|_{\omega,\Omega} \leq \max(1, \mathscr{C}_{\rm ap}) \eta.$$

Approximation factor

$$\mathscr{C}_{\rm ap} := \max_{\substack{g \in L^2(\Omega) \\ \|g\|_{\mu,\Omega}=1}} \min_{v_h \in V_h} \omega \, |\!|\!| \mathscr{S} g - v_h |\!|\!|_{\omega,\Omega} \to 0.$$

T. Chaumont-Frelet, A. Ern and M. Vohralík, *Numer. Math.*, 2021.

W. Dörfler, S. Sauter, *Comput. Meth. Appl. Math.*, 2013.

High frequencies

Takeaways

Takeways

The "equilibration" technology is the same than for low frequencies.

Corse error estimate

$$|||e_h|||_{\omega,\Omega} \leq \max(1, \mathscr{C}_{\rm st})\eta \quad \mathscr{C}_{\rm st} \gtrsim \omega\ell_\Omega/c_{\min}$$

Sharp error estimate

$$|||e_h|||_{\omega,\Omega} \leq \max(1, \mathscr{C}_{\rm ap})\eta \quad \mathscr{C}_{\rm ap} \to 0$$

Efficiency

$$\eta \leq C_{\rm eff}\left(1 + \frac{\omega h}{c_{\min} p}\right)|||e_h|||_{\omega,\Omega}$$

Controlling the pre-factors

Controlling the pre-factors

The stability constant $\mathscr{C}_{\mathrm{st}}$

The stability constant

The stability constant is defined by

$$\mathscr{C}_{\mathrm{st}} := \max_{\substack{g \in L^2(\Omega) \\ \|g\|_{\mu,\Omega}=1}} \|\nabla(\mathscr{S}^\star g)\|_{\mathbf{A},\Omega}.$$

It is only related to the PDE, and independent of the numerical scheme.

Qualitative behaviour

It is known that we have "at least":

$$\mathscr{C}_{\rm st} \gtrsim \frac{\omega \ell_\Omega}{c_{\min}}.$$

For non-trapping settings (the "easier" scenario), we have

$$\mathscr{C}_{\rm st} \lesssim \frac{\omega \ell_\Omega}{c_{\min}}.$$

If strong traping happens, "extreme" behaviors can occur

$$\mathscr{C}_{\rm st} \gtrsim \exp\left(\alpha \frac{\omega \ell_\Omega}{c_{\min}}\right)$$

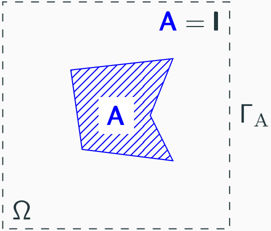
for "some" frequencies. For "most frequency"

$$\mathscr{C}_{\rm st} \gtrsim \left(\frac{\omega \ell_\Omega}{c_{\min}}\right)^\beta.$$

D. Lafontaine, E.A. Spence and J. Wunsch, *Comm. Pure Appl. Math.* 2021

Quantitative estimate for star-shaped non-trapping obstacles

$\Omega := (-\ell/2, \ell/2)^3$ is a cube centered at the origin.
$D \subset \Omega$ is star shaped with respect to the origin.



Assume that $\gamma = 1$ and that
$\mu = 1$ and $\mathbf{A} = \boldsymbol{I}$ in $\Omega \setminus D$.

Assume that $\mu = \mu_D \geq 1$ and $\mathbf{A} = \mathbf{A}_D \preceq \boldsymbol{I}$ in D.

This discribes an obstacle made of a material with a "slow" waveseed.

Quantitative estimate for star-shaped non-trapping obstacles

Guaranteed upper bound

$$\mathscr{C}_{\mathrm{st}} \leq 6 + \frac{3+\sqrt{3}}{\sqrt{3}} \frac{\omega \ell_\Omega}{c_{\min}}$$

The proof relies on a "Morawetz multiplier":
multiply the PDE by $\boldsymbol{x} \cdot \boldsymbol{\nabla} u$ and integrate by parts until it works!

- C.S. Morawetz, *Comm. Pure Appl. Math.*, 1962.
- J.M. Melenk, *PhD thesis*, 1995.
- H. Barucq, T. Chaumont-Frelet and C. Gout, *Math. Comp.*, 2016.
- T. Chaumont-Frelet, A. Ern, E. Vohralìk, *Numer. Math.* 2021.

Controlling the pre-factors

The approximation factor $\mathscr{C}_{\mathrm{ap}}$

The approximation factor

The approximation factor is defined by

$$\mathscr{C}_{\mathrm{ap}} := \max_{\substack{g \in L^2(\Omega) \\ \|g\|_{\mu,\Omega}=1}} \min_{v_h^\star \in V_h} \|\boldsymbol{\nabla}(\mathscr{S}^\star g - v_h^\star)\|_{\mathbf{A},\Omega}.$$

It depends on both the PDE and the approximation space V_h.

Assuming that $\mathbf{A} = \boldsymbol{I}$, Ω is convex, and $\mathscr{C}_{\mathrm{st}}$ is known, we can control it.

Idea one: explicit interpolation error

R. Arcangeli and J.L. Gout, RAIRO Numer. Anal., 1976.

If $v \in H^2(\Omega)$, let $I_h^1 v \in V_h$ denotes its first-order Lagrange interpolant:

$$\|\nabla(v - I_h^1 v)\|_{\mathbf{A},\Omega} \leq \mathscr{C}_{\mathcal{T},\mathrm{i}} h \|\nabla^2 v\|_{\Omega},$$

with a constant $\mathscr{C}_{\mathcal{T},\mathrm{i}}$ that is easily computable.

We then have

$$\begin{aligned}
\mathscr{C}_{\mathrm{ap}} &:= \max_{\substack{g \in L^2(\Omega) \\ \|g\|_{\mu,\Omega}=1}} \min_{v_h^\star \in V_h} \|\nabla(\mathscr{S}^\star g - v_h^\star)\|_{\mathbf{A},\Omega} \\
&\leq \max_{\substack{g \in L^2(\Omega) \\ \|g\|_{\mu,\Omega}=1}} \|\nabla(\mathscr{S}^\star g - I_h^1(\mathscr{S}^\star g)\|_{\mathbf{A},\Omega} \\
&\leq \mathscr{C}_{\mathcal{T},\mathrm{i}} h \max_{\substack{g \in L^2(\Omega) \\ \|g\|_{\mu,\Omega}=1}} \|\nabla^2(\mathscr{S}^\star g)\|_{\Omega}.
\end{aligned}$$

Idea two: estimation of the Hessian norm

- P. Grisvard, 1985.
- T. Chaumont-Frelet, S. Nicaise and J. Tomezyk, *Comm. Pure Appl. Anal.*, 2020.

Because Ω is convex and $\gamma = 1$, we have

$$\|\nabla^2(\mathscr{S}^\star g)\|_\Omega \leq \|\Delta(\mathscr{S}^\star g)\|_\Omega.$$

Then, we use the facts that

$$-\Delta(\mathscr{S}^\star g) = 2\mu\omega^2 g + \mu\omega^2 \mathscr{S}^\star g$$

and

$$\omega\|\mathscr{S}^\star g\|_{\mu,\Omega} \leq 2\mathscr{C}_{\rm st}\|g\|_{\mu,\Omega},$$

to show that

$$\|\nabla^2(\mathscr{S}^\star g)\|_\Omega \leq 2\frac{\omega}{c_{\min}}\left(1+\mathscr{C}_{\rm st}\right)\|g\|_{\mu,\Omega}.$$

Explicit control of the approximation factor

Recall that

$$\mathscr{C}_{\rm ap} \leq \mathscr{C}_{\mathcal{T},{\rm i}} h \max_{\substack{g \in L^2(\Omega) \\ \|g\|_{\mu,\Omega}=1}} \|\boldsymbol{\nabla}^2(\mathscr{S}^\star g)\|_\Omega$$

and

$$\|\boldsymbol{\nabla}^2(\mathscr{S}^\star g)\|_\Omega \leq 2\frac{\omega}{c_{\min}}\left(1+\mathscr{C}_{\rm st}\right)\|g\|_{\mu,\Omega} \quad \forall g \in L^2(\Omega).$$

Guaranteed bound

$$\mathscr{C}_{\rm ap} \leq 2\left(1+\mathscr{C}_{\mathcal{T},{\rm i}}\right)\frac{\omega h}{c_{\min}}\mathscr{C}_{\rm st}$$

Controlling the pre-factors

Takeaways

Takeaways

The estimator η needs to be "pre-factored" by $\mathscr{C}_{\mathrm{st}}$ or $\mathscr{C}_{\mathrm{ap}}$.
The "qualitative" behaviors of both quantities are relatively well known.

The behaviour of $\mathscr{C}_{\mathrm{st}}$ is only dictated by the PDE.
Explicit bounds are available for non-trapping star-shaped obstables.

The approximation factor $\mathscr{C}_{\mathrm{ap}}$ depends on the PDE and V_h.
When $\mathbf{A} = \boldsymbol{I}$, Ω is convex and $\mathscr{C}_{\mathrm{st}}$ is known, we can bound it nicely.

Numerical illustrations

Numerical illustrations

A validation experiment

Propagation of a plane wave

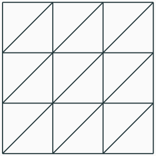
We consider the propagation of a plane wave in $\Omega = (-1, 1)^2$

$$\begin{cases} -\omega^2 u - \Delta u &= 0 \quad \text{in } \Omega, \\ \nabla u \cdot \boldsymbol{n} - i\omega u &= g \quad \text{on } \Gamma_{\mathrm{A}}, \end{cases}$$

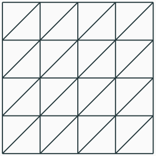
where

$$g := \nabla \xi_\theta \cdot \boldsymbol{n} - i\omega\xi_\theta \quad \xi_\theta := e^{i\omega \boldsymbol{d}\cdot\boldsymbol{x}}$$

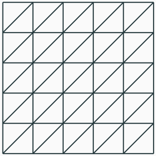
with $\boldsymbol{d} := (\cos\theta, \sin\theta)$ and $\theta = \pi/12$. The solution is $u = \xi_\theta$.



$h = \sqrt{2} \times 2/3$

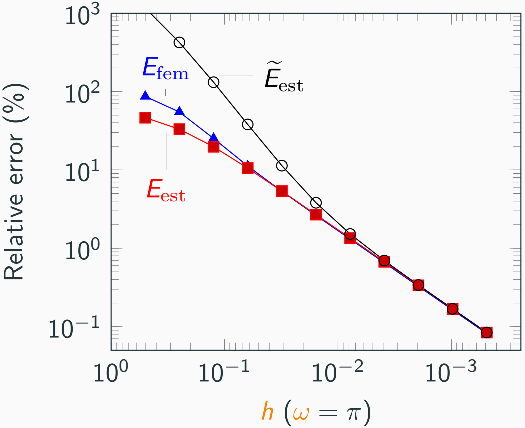


$h = \sqrt{2} \times 1/2$



$h = \sqrt{2} \times 2/5$

Plane wave experiment $p = 1$ and $\omega = \pi$

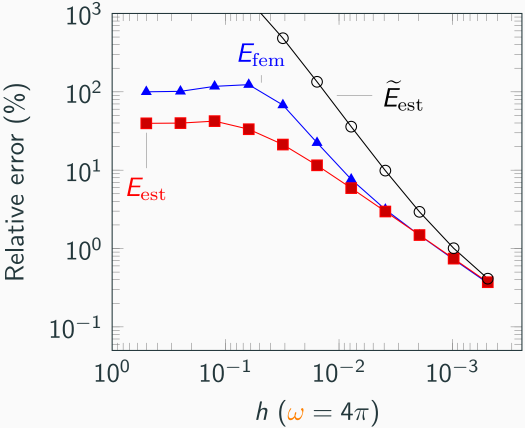


$E_{\text{fem}} := \|\!| e_h |\!\|_{\omega,\Omega}$

$E_{\text{est}} := \eta$

$\widetilde{E}_{\text{est}} := (1 + \mathscr{C}_{\text{ap}})\eta$

Plane wave experiment $p = 1$ and $\omega = 4\pi$

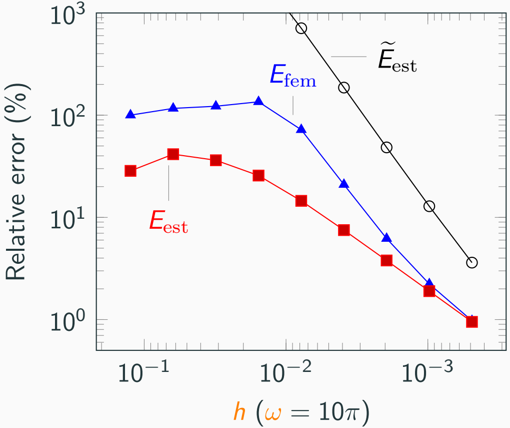


$E_{\mathrm{fem}} := \|e_h\|_{\omega,\Omega}$

$E_{\mathrm{est}} := \eta$

$\widetilde{E}_{\mathrm{est}} := (1 + \mathscr{C}_{\mathrm{ap}})\eta$

Plane wave experiment $p = 1$ and $\omega = 10\pi$

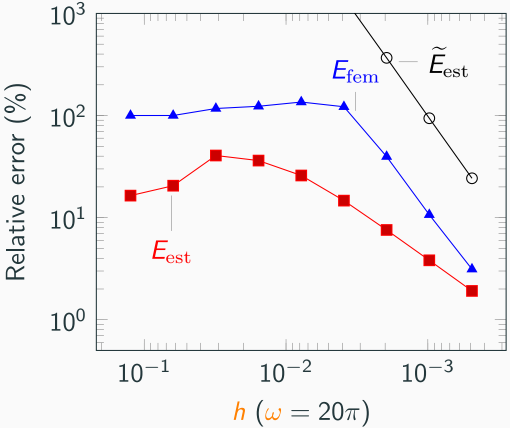


$E_{\rm fem} := \|e_h\|_{\omega,\Omega}$

$E_{\rm est} := \eta$

$\widetilde{E}_{\rm est} := (1 + \mathscr{C}_{\rm ap})\eta$

Plane wave experiment $p = 1$ and $\omega = 20\pi$

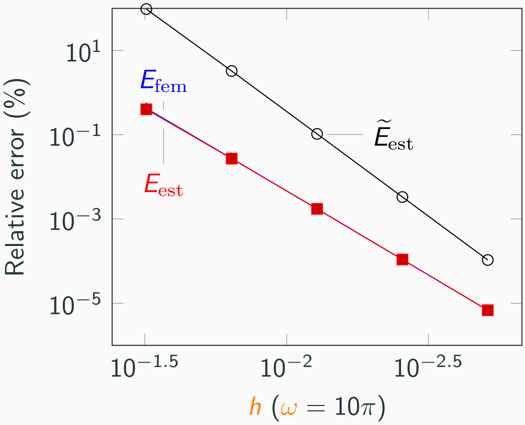


$E_{\text{fem}} := \|e_h\|_{\omega,\Omega}$

$E_{\text{est}} := \eta$

$\widetilde{E}_{\text{est}} := (1 + \mathscr{C}_{\text{ap}})\eta$

Plane wave experiment $p = 4$ and $\omega = 10\pi$

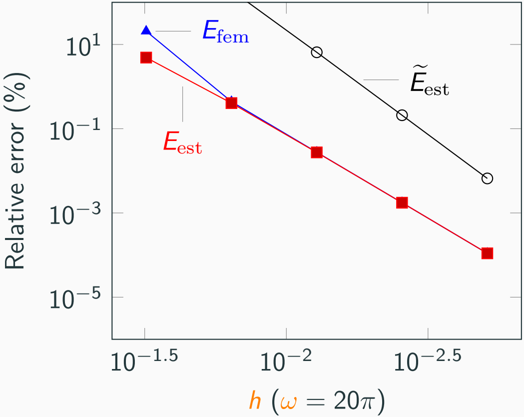


$E_{\mathrm{fem}} := \|e_h\|_{\omega,\Omega}$

$E_{\mathrm{est}} := \eta$

$\widetilde{E}_{\mathrm{est}} := (1 + \mathscr{C}_{\mathrm{ap}})\eta$

Plane wave experiment $p = 4$ and $\omega = 20\pi$

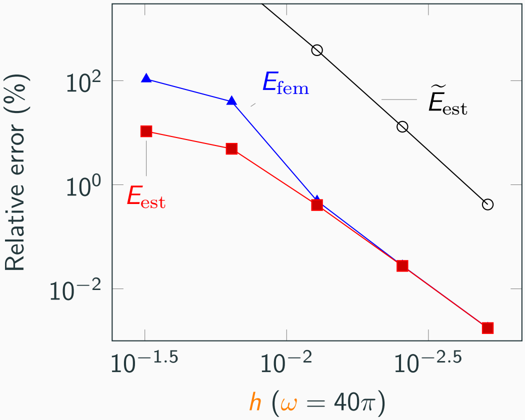


$E_{\mathrm{fem}} := \|e_h\|_{\omega,\Omega}$

$E_{\mathrm{est}} := \eta$

$\widetilde{E}_{\mathrm{est}} := (1 + \mathscr{C}_{\mathrm{ap}})\eta$

Plane wave experiment $p = 4$ and $\omega = 40\pi$

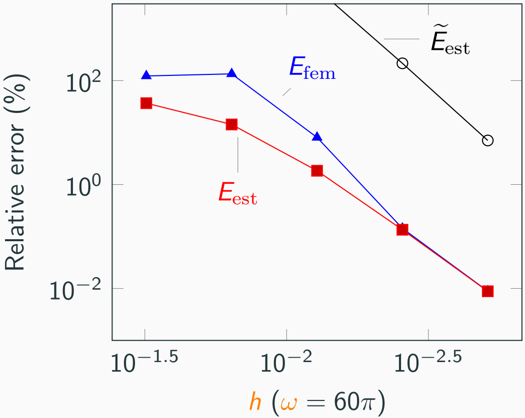


$E_{\mathrm{fem}} := \|e_h\|_{\omega,\Omega}$

$E_{\mathrm{est}} := \eta$

$\widetilde{E}_{\mathrm{est}} := (1 + \mathscr{C}_{\mathrm{ap}})\eta$

Plane wave experiment $p = 4$ and $\omega = 60\pi$



$E_{\mathrm{fem}} := \|e_h\|_{\omega,\Omega}$

$E_{\mathrm{est}} := \eta$

$\widetilde{E}_{\mathrm{est}} := (1 + \mathscr{C}_{\mathrm{ap}})\eta$

Numerical illustrations

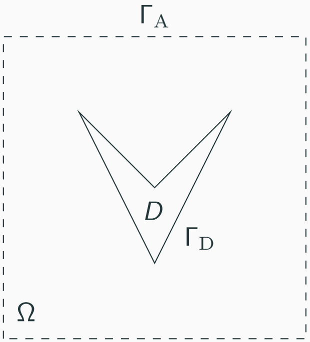
A more realistic example

Scattering by an non-trapping obstacle

We now consider a scattering problem

$$\begin{cases} -\omega^2 u - \Delta u &= 0 & \text{in } \Omega, \\ u &= 0 & \text{on } \Gamma_{\mathrm{D}}, \\ \nabla u \cdot \boldsymbol{n} - i\omega u &= g & \text{on } \Gamma_{\mathrm{A}}, \end{cases}$$

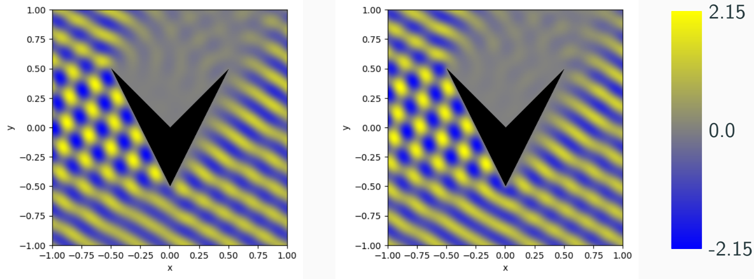
where again $g = \nabla \xi_\theta \cdot \boldsymbol{n} - i\omega\xi_\theta$.



We fix the wavenumber $\omega = 10\pi$ and employ $\mathbb{P}_3$ elements.

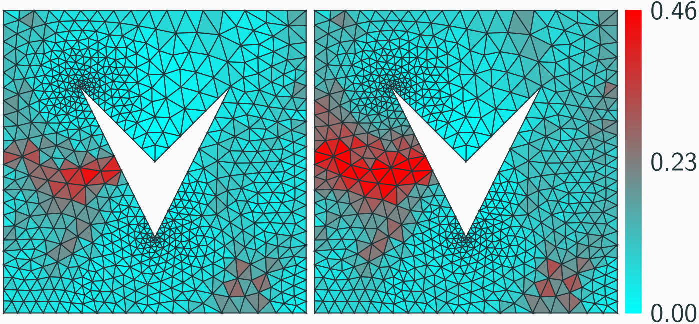
We consider a sequence of meshes that are adaptively refined using η_K.

Solution of the scattering problem



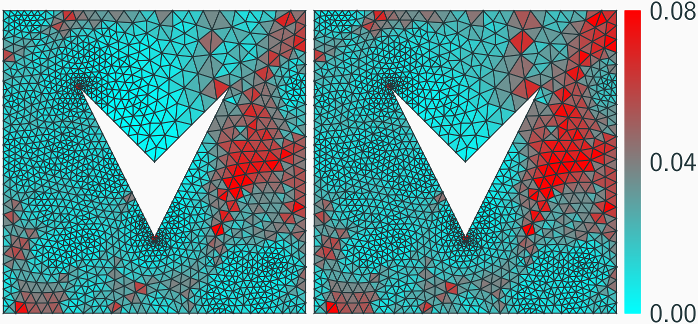
Real (left) and imaginary (right) parts of the solution

Estimated error in mesh #1



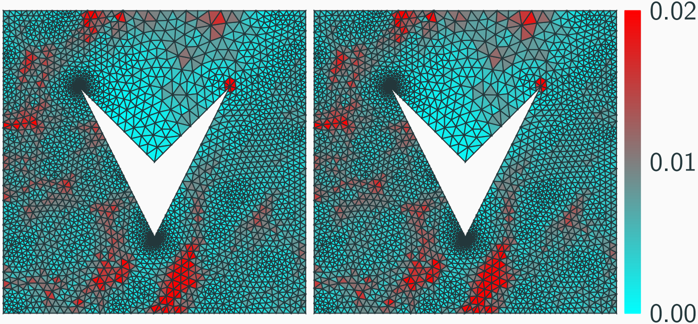
Estimator η_K (left) and elementwise error $|||e_h|||_{\omega,K}$ (right)

Estimated error in mesh #2



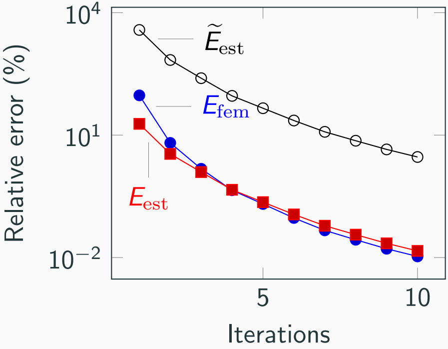
Estimator η_K (left) and elementwise error $|||e_h|||_{\omega,K}$ (right)

Estimated error in mesh #3



Estimator η_K (left) and elementwise error $\|e_h\|_{\omega,K}$ (right)

Behavior of the estimator through the adaptive procedure



$$E_{\mathrm{fem}} := \| e_h \|_{\omega,\Omega}$$

$$E_{\mathrm{est}} := \eta$$

$$\widetilde{E}_{\mathrm{est}} := (1 + \mathscr{C}_{\mathrm{st}})\eta$$

Behaviors of the estimated and analytical errors in the adaptive procedure

Concluding remarks

Concluding remarks

Takeaways

Takeaways

We construct an a posteriori error estimator η *via* flux equilibration.
It directly provides guaranteed error estimates at low frequencies.

For high frequencies, η has to be pre-factored, either by $\mathscr{C}_{\mathrm{st}}$ and $\mathscr{C}_{\mathrm{ap}}$.
The estimates are asymptotically constant-free.
In specific situations, we can provide guaranteed bounds on $\mathscr{C}_{\mathrm{st}}$ and $\mathscr{C}_{\mathrm{ap}}$.

There is still a long way toward fully reliable error estimation for high-frequency problems!

T. Chaumont-Frelet, A. Ern and M. Vohralìk, *Numer. Math.*, 2021.

Concluding remarks

Extensions

Extensions

We can obtain guaranteed bounds on $\mathscr{C}_{\mathrm{st}}$ in weakly trapping geometry with "directional" Morawetz mutliplier

$$(\boldsymbol{x}_d \boldsymbol{e}^d) \cdot \boldsymbol{\nabla} u.$$

S.N. Chandler-Wilde et. al., *SIAM J. Math. Anal.*, 2020.

T. Chaumont-Frelet and E.A. Spence, *submitted*.

T. Chaumont-Frelet and Z. Kassali, *in progress*.

It is possible to obtain approximations for $\mathscr{C}_{\mathrm{ap}}$ in more general situations.

Extensions

Everything I presented (painfully) extends to Maxwell's equations:

- T. Chaumont-Frelet, A. Ern and M. Vohralìk, *C.R. Math. Acad. Sci.*, 2020.
- T. Chaumont-Frelet, A. Ern and M. Vohralìk, *Math. Comp.*, 2021.
- T. Chaumont-Frelet and M. Vohralìk, *submitted*.
- T. Chaumont-Frelet, *will be submitted soon*!

Thanks for your attention! :-)