Eigenvalue and resonance asymptotics in perturbed waveguide: twisting versus bending

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Plan

1. Some fibered operators and their band functions
2. The Laplacian in a deformed waveguide
3. Main results: asymptotics of the resonance
4. Proofs and generalizations
Fibered operators

Diagonalization of an operator:

- Models with invariance properties are reduced to lower dimensional problems.
  - Translation invariance: partial Fourier transform:
    - Magnetic field, waveguides, stratified media.
  - Periodic invariance: Floquet-Bloch transform.
    - Cristallin structures, Graphene.

- General framework: an operator $H_0$ with invariance properties writes

$$UH_0U^* = \int \bigoplus h_0(p)dp, \text{ with } U \text{ a unitary transform.}$$

- Fiber operators $h_0(p)$ (may) have a discrete sectrum $(E_n(p))_{n \geq 1}$.

Spectrum of $H_0$:

- The band functions are $p \mapsto E_n(p)_{n \geq 1}$. The spectrum of $H_0$ is

$$\sigma(H_0) = \bigcup_{n \geq 1} \text{Ran } E_n.$$

- Non constant band functions correspond to absolutely continuous spectrum.
Some questions (among others)

Transport properties

- Consider the Schrödinger equation

\[ i \partial_t \psi = H_0 \psi. \]

- Spectral analysis of \( H_0 \) required for the time dependant analysis.
- Gaps in the spectrum correspond to the absence of propagation in the direction of invariance.
- After perturbation, discrete eigenvalues correspond to trapped modes.
- Resonances play a role in scattering theory.

Spectral analysis

- Critical points of the band functions are thresholds in the spectrum of \( H_0 \).
  General theory in [GeNi98] for proper analytical band functions.
- They correspond to instable energies of the system.
  → No standard limiting absorption principle.
  → More eigenvalues and resonances after perturbation.
Somes examples: The magnetic Laplacian

Constant magnetic field $B = 1$ in a half-plane $\mathbb{R}^2_+ = \{(s, t) \in \mathbb{R} \times \mathbb{R}_+\}$. Classical trajectories:
Some fibered operators and their band functions

Somes examples: The magnetic Laplacian

The Schrödinger operator:

- The magnetic potential \( A(s, t) = (-t, 0) \) generates a magnetic field
  \[
  B = \text{curl } A = 1.
  \]

- The magnetic Laplacian is (the closure of)
  \[
  H_0 = (-i\nabla - A)^2 = -\partial_t^2 + (-i\partial_s - t)^2 \text{ in } L^2(\mathbb{R}_+^2)
  \]
  with (let’s say) Neumann boundary conditions.

Reduction of dimension:

- Here, \( U \) is the partial Fourier transform in \( s \). The fiber operator is for \( p \in \mathbb{R} \):
  \[
  h_0(p) = -\partial_t^2 + (t - p)^2 \text{ in } L^2(\mathbb{R}_+).
  \]

- The band functions are the values \( E \) for which there exists \( y \neq 0 \) solution of
  \[
  \begin{cases}
  -y''(t) + (t - p)^2 y(t) = Ey(t), & t > 0, \\
  y'(0) = 0.
  \end{cases}
  \]
Somes examples: The magnetic Laplacian

Each band function $E_n$ has a unique non-degenerate minimum ([Dg,DH93]).

\[ \sigma(H_0) = [\min E_1, +\infty). \]

It also tends to a finite limit (Landau levels)!

Figure: Abscissa: $p$ (Fourier parameter dual to $s$).
More magnetic models: the snake’s orbits

\[ B = 1 \]

\[ B = a > 1 \]

Figure: Classical orbits of an electron submitted to magnetic steps
More magnetic models: the snake’s orbits

\[ B = 1 \]

\[ B = -1 \]

Figure: Classical orbits of an electron submitted to magnetic steps
More magnetic models: the snake’s orbits

Band functions for the symmetric magnetic step:

![Graph showing band functions for the symmetric magnetic step.](image)
Some fibered operators and their band functions

Some examples: The Laplacian in a perturbed waveguide

To be continued
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Construction of the waveguide

Geometric data of the waveguide:

- A reference curve $\gamma : \mathbb{R} \to \mathbb{R}^3$, defined by a curvature $\kappa$ and a torsion.
- A cross section $\omega \subset \mathbb{R}^2$, an open bounded lipschitz domain.
- A rotation $\theta : \mathbb{R} \to \mathbb{R}$ of the section around $\gamma$.

To make it simpler, assume that the torsion of $\gamma$ is 0. The rotation $\theta$ will play a similar role...
Tubular coordinate

Construction of a waveguide $\Omega \subset \mathbb{R}^3$:

- Let $(e_1, e_2, e_3)$ be a Frenet frame associated with $\gamma$.
- Let $e_2^\theta(s), e_3^\theta(s)$ be the frame obtained by applying a rotation of angle $\theta(s)$ to $(e_2(s), e_3(s))$ around $e_1(s)$.
- Define $\mathcal{L} : \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}^3$ by

$$\mathcal{L}(s, t_2, t_3) = \gamma(s) + t_2 e_2^\theta(s) + t_3 e_3^\theta(s)$$

With additional hypotheses, $\mathcal{L}$ is a diffeomorphism on $\mathbb{R} \times \omega$, and the waveguide is defined by

$$\Omega := \mathcal{L}(\mathbb{R} \times \omega).$$
The reference unperturbed operator

The reference tube $\Omega_0$:

- The curve $\gamma(\mathbb{R})$ is a line (i.e. $\kappa = 0$).
- The twisting $\theta'$ is a constant $\beta$, with two different case:
  - $\beta = 0$: straight tube (a cylinder)
  - $\beta \neq 0$: periodically twisted tube.
The reference unperturbed operator

Our reference operator:

The Laplacian $-\Delta$ in $\Omega_0$ with Dirichlet boundary conditions.

Denote $\varphi$ the cylindrical variable and $\partial_\varphi = t_2 \partial_3 - t_3 \partial_2$ the angular derivative.

After a change of variable the original Laplacian is unitarily equivalent to

$$H_0 = -\partial_2^2 - \partial_3^2 - (\partial_s - \beta \partial_\varphi)^2 \quad \text{in} \quad L^2(\mathbb{R} \times \omega) \quad \text{with Dirichlet b.c.}$$

Let $\mathcal{F}_s$ be the Partial Fourier transform in the $s$ variable.

Fiber decomposition:

$$\mathcal{F}_s H_0 \mathcal{F}_s^* = \int_{p \in \mathbb{R}} h_0(p) dp.$$

$$h_0(p) := -\Delta_\omega - (-ip - \beta \partial_\varphi)^2 \quad \text{in} \quad L^2(\omega) \quad \text{with Dirichlet b.c.}$$
The operators \((h_0(p))_{p \in \mathbb{R}}\) form a type A analytic family of self-adjoint operators with compact resolvent.
Denote by \((E_n(p))_{n \geq 0}\) the increasing sequence of eigenvalues of \(h_0(p)\). Then
\[
\sigma(H_0) = \bigcup_{n \geq 1} E_n(\mathbb{R}) = [\mathcal{E}_1, +\infty) \quad \text{with} \quad \mathcal{E}_n = \min_{p \in \mathbb{R}} E_n(p).
\]

Analysis of the first band function, done in [Briet-Kovarik-Raikov-Soccorsi 08]:
- The first eigenvalue \(E_1(p)\) is non-degenerate (simple).
- \(\mathcal{E}_1 = E_1(0)\) and this minimum is non-degenerate and unique.

The case \(\beta = 0\) of a straight tube
- In the cylinder \(\mathbb{R} \times \omega\), the variables decouple:
  \[
  -\Delta_{\Omega_0} = \text{Id} \otimes (-\Delta_\omega) + (-\partial_s^2) \otimes \text{Id}
  \]
- The band functions are explicit:
  \[
  E_n(p) = \mathcal{E}_n + p^2
  \]
  here \((\mathcal{E}_n)_{n \geq 1}\) are the eigenvalues of \(-\Delta_\omega\) (with Dirichlet b.c.).
Perturbation of the waveguide

Let $\kappa$ and $\tau$ be two real functions such that

$$\lim_{\pm \infty} \kappa = \lim_{\pm \infty} \tau = 0.$$ 

Consider $\Omega = \mathcal{L}(\mathbb{R} \times \omega)$, the tube of cross section $\omega$, along the curve $\gamma$ and twisted by $\theta$ with $\theta' = \beta + \tau$.

Let $H$ be the Dirichlet Laplacian in $\Omega$. The metric is $G := (d\mathcal{L})^T(d\mathcal{L})$.

$$G = \begin{pmatrix}
    h^2 + h_2^2 + h_3^2 & h_2 & h_3 \\
    h_2 & 1 & 0 \\
    h_3 & 0 & 1
\end{pmatrix}$$

with

$$
\begin{cases}
    h(s, t) = 1 - \kappa(s)(t_2 \cos \theta(s) + t_3 \sin \theta(s)) \\
    h_2(s, t) = t_3 \theta'(s) \\
    h_3(s, t) = -t_2 \theta'(s)
\end{cases}
$$

Remark that $h = \sqrt{\det(G)}$, and note $G^{-1} = G^{jk}$. The operator is

$$-\Delta_\Omega \equiv \frac{1}{h} \sum_{j,k=1}^3 \partial_j h G^{jk} \partial_k$$

on $L^2(\mathbb{R} \times \omega, h)$ with Dirichlet b.c.
Creation of eigenvalues

After a change of variables:

\[
H \equiv -\partial_2^2 - \partial_3^2 - \frac{\kappa^2}{4h^2} - \left( h^{-1/2} (\partial_s - \theta' \partial_\varphi) h^{-1/2} \right)^2 \text{ in } L^2(\mathbb{R} \times \omega) \text{ with Dirichlet b.c.}
\]

Its essential spectrum is still \([\mathcal{E}_1, +\infty)\). What about discrete spectrum?
Twisting vs bending: references

(Some) Known results for $\beta = 0$

- Pure bending ($\theta' = 0$ and $\kappa \neq 0$) creates discrete eigenvalues (Duclos-Exner 95, Grushin 04).
- Pure twisting ($\kappa = 0$ and $\tau \neq 0$) does not change the spectrum (Grushin 04). Existence of a Hardy inequality (Ekholm-Kovarik-Krejcirik 08) proves that adding a small bending ($\kappa \ll 1$) does not add discrete spectrum.

(Some) Known results for $\beta > 0$ and $\kappa = 0$.

- Small enhanced twisting $0 < \tau \ll 1$ does not change the spectrum ([Briet-Hammadi-Krejcirik 15])
- Slowed twisting ($\int_{\tau} < 0$) creates eigenvalues (eventually an infinite number). Counting function studied in Briet-Kovarik-Raikov-Soccorsi 08].
- Scattering properties in Briet-Kovarik-Raikov 15].
Our perturbative approach

Some questions:
- No result when $\beta \neq 0$ and $\kappa \neq 0$.
- For straight tubes, switching from bending to twisting, eigenvalue(s) disappear in the essential spectrum. Are they resonances?
- Provide quantitative criterion to compare twisting and bending, even when $\beta \neq 0$.

Our approach:
- Consider a waveguide along a curve with curvature $\delta \kappa$, with cross section $\omega$ fixed, and twisted by a rotating function $\theta' = \beta + \delta \tau$.
- Study what happen near $\mathcal{E}_1$ as $\delta \to 0$

Recall that the reference operator is

$$H_0 = -\partial_2^2 - \partial_3^2 - (\partial_s - \beta \partial_\varphi)^2.$$

The perturbed operator is

$$H_\delta := -\partial_2^2 - \partial_3^2 - \delta^2 \frac{\kappa^2}{4 h_\delta^2} - \left(h_\delta^{-1/2} (\partial_s - (\beta + \delta \tau) \partial_\varphi) h_\delta^{-1/2} \right)^2$$

with $h_\delta(s, t) = 1 - \delta \kappa(s)(t_2 \cos \theta_\delta(s) + t_3 \sin \theta_\delta(s))$, $\theta'_\delta = \beta + \delta \tau$. 
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Recall the meromorphic extension for the 1d Laplacian

- $z \mapsto (-\Delta_{\mathbb{R}} - z)^{-1}$ is well defined on $\mathbb{C} \setminus [0, +\infty)$.
- Set $z = k^2$, with $k \in \mathbb{C}^+ = \{ \text{im}(k) > 0 \}$. Then $(-\Delta - k^2)^{-1}$ has a kernel

$$\forall k \in \mathbb{C}^+, \quad \int_{\mathbb{R}} \frac{e^{ip(s-s')}}{p^2 - k^2} dp = \frac{e^{ik|s-s'|}}{2ik}.$$ 

**Theorem**

Let $w(s) = \exp(-a|s|)$ with $a > 0$. Then $k \mapsto (-\Delta_{\mathbb{R}} - k^2)^{-1}$, acting on $wL^2(\mathbb{R})$, initially defined on $\mathbb{C}^+$, admits a meromorphic extension to $\{ \text{im}(k) > -a \}$, with a unique pole in 0.
Resonances for perturbations

- For suitable $V$, the resolvent $(-\Delta_R + V - k^2)^{-1}$ is meromorphic on a neighborhood of 0.
- Its poles are the resonances of $-\Delta_R + V$.
- For small $V$, you expect a single resonance near 0.
- When $k \in \mathbb{C}^+ \cap i\mathbb{R}$ is a resonance, it corresponds to a negative eigenvalue.

\[ z \in \mathbb{C} \setminus [0, +\infty) \]
\[ \sigma(H_0) \]
\[ 0 \]

Motivations for finding resonances

- Expansion of the semi-group: near an isolated resonance $k_0$ for $H$,
  \[ e^{-itH} \approx e^{-itz_0} \Pi + R \]
  with \( \{ \Pi \) rank one projector
  \( R \) remainder
- Singularities of the spectral shift function and Breit-Wigner formulas.
- Poles of the scattering matrix
Resolvent of our free operator

Lemma

The free resolvent \( \mathbb{C}^+ \ni k \mapsto (H_0 - \mathcal{E}_1 - k^2)^{-1} \), acting on the weighted space \( wL^2(\mathbb{R} \times \omega) \), admits a meromorphic extension near 0.

- **Proof:** extend singular Cauchy integrals of the form

\[
\int_{\mathbb{R}} e^{ip(s-s')} \psi_n(t, p)\psi_n(t', p)(E_n(p) - \mathcal{E}_1 - k^2)^{-1} dp,
\]

- Similar results near other non degenerate critical points of the band functions, in particular near each \( \mu_n \) when \( \beta = 0 \).
- General method for analytically fibered operators, see [Gérard 90].

**Hypotheses on the waveguide:**

- The functions \( \kappa \) and \( \tau \) are \( C^2 \).
- These functions, their first and second derivative satisfy

\[
\kappa(s), \tau(s) = O(e^{-\alpha s^2})
\]

for some \( \alpha > 0 \).
Main result for periodically twisted waveguide

**Theorem (B.M.P.P 2018)**

Fix a sufficiently small neighborhood of zero $\mathcal{D}$ in $\mathbb{C}$. Then, there exists $\delta_0 > 0$ such that for $\delta \leq \delta_0$, the function $\mathbb{C}^+ \ni k \mapsto (H_\delta - E_1 - k^2)^{-1}$, acting on $wL^2(\mathbb{R} \times \omega)$, admits a meromorphic extension on $\mathcal{D}$. This function has a unique pole $k(\delta)$ in $\mathcal{D}$. It has multiplicity one and satisfies

$$k(\delta) = i\mu_1\delta + O(\delta^2), \quad \mu_1 \in \mathbb{R}.$$ 

Moreover, there exists a function $F : \mathbb{R} \to \mathbb{R}$, constants $c_j > 0$, such that

$$\mu_1 = -c_1\beta \int_{\mathbb{R}} \tau(s)ds + c_2\beta^2 \int_{\mathbb{R}} \kappa(s)F(s)ds.$$ 

The function $F$, $c_1$ and $c_2$ depend only on $\beta$ and $\omega$ (explicit).

Further, the pole $k(\delta)$ is a purely imaginary number.
Main results: asymptotics of the resonance.

Main result for periodically twisted waveguide

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The function $F$, $c_1$ and $c_2$ depend only on $\beta$ and $\omega$ (explicit).

Further, the pole $k(\delta)$ is a purely imaginary number.

$$F(s) = \int_{\omega} \left( |\partial_\varphi \psi_1(t)|^2 + \frac{1}{4} |\psi_1(t)|^2 \right) \left( t_2 \cos(\beta s) + t_3 \sin(\beta s) \right) dt,$$
Comments on the result

Recall that our resonance of $H_\delta - E_1$ satisfies

$$k(\delta) = i\mu_1 \delta + O(\delta^2), \quad \mu_1 \in \mathbb{R}.$$  

with

$$\mu_1 = -c_1 \beta \int_\mathbb{R} \tau(s) ds + c_2 \beta^2 \int_\mathbb{R} \kappa(s) F(s) ds.$$  

Localization of the resonance:

- When $\mu_1 > 0$,
  
  $$E_1 + k(\delta)^2 = E_1 - \delta^2 \mu_1^2 + O(\delta^3)$$  
  is a discrete eigenvalue below $E_1$.

- When $\mu_1 < 0$, it gives an antiboundstate.

- When $\mu_1 = 0$, you need to go to the next order (hard).

Influence of the geometry:

- When $\kappa = 0$, it depends on the sign of $\int_\mathbb{R} \tau$.

- When $\int_\mathbb{R} \tau = 0$, both cases can appear.

- When $\beta \ll 1$, you can focus on the first term.

- When $\beta = 0$, you need to go to the next order.
Main results: asymptotics of the resonance.

Main result for perturbation of a straight waveguide

Theorem (B.M.P.P 2018)

Assume $\beta = 0$, the same results hold, but $k(\delta)$ satisfies

$$k(\delta) = i\mu_2 \delta^2 + O(\delta^3), \quad \mu_2 \in \mathbb{R}.\,$$

Moreover, there exist positive bilinear form $q_1$, $q_2$, and a bilinear form $q_3$ such that

$$\mu_2 = q_1(\kappa, \kappa) - q_2(\tau, \tau) + q_3(\tau, \dot{\kappa})$$

Similar result near each threshold $E_n$, depending on its multiplicity as an eigenvalues of $-\Delta_\omega$. 

Main result for perturbation of a straight waveguide

Theorem (B.M.P.P 2018)

Assume $\beta = 0$, the same results hold, but $k(\delta)$ satisfies

$$k(\delta) = i\mu_2 \delta^2 + O(\delta^3), \quad \mu_2 \in \mathbb{R}.$$ 

Moreover, there exist positive bilinear form $q_1, q_2,$ and a bilinear form $q_3$ such that

$$\mu_2 = q_1(\kappa, \kappa) - q_2(\tau, \tau) + q_3(\tau, \dot{k})$$

$$\mu_2 = \frac{1}{8} \sum_{q \geq 2} (E_q - E_1)^2 \langle \psi_q | t_2 \psi_1 \rangle^2 \langle \kappa | (-\partial_s^2 + E_q - E_1)^{-1} \kappa \rangle$$

$$- \frac{1}{2} \sum_{q \geq 2} (E_q - E_1) \langle \psi_q | \partial_\varphi \psi_1 \rangle^2 \langle \tau | (-\partial_s^2 + E_q - E_1)^{-1} \tau \rangle$$

$$+ \frac{1}{2} \sum_{q \geq 2} (E_q - E_1) \langle \psi_q | \partial_\varphi \psi_1 \rangle \langle \psi_q | t_2 \psi_1 \rangle \langle \tau | (-\partial_s^2 + E_q - E_1)^{-1} \dot{k} \rangle.$$
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Birman Schwinger form and Grushin matrix:

Main ingredients:

- Laurent expansion of the free resolvent:

\[ R_0(k) = (H_0 - \epsilon_1 - k^2)^{-1} = \frac{A_{-1}}{k} + F(k) \text{ with } F \text{ holomorphic near } 0 \]

- Resolvent identity and Birman Schwinger principle led to invert

\[ I + (I + \delta V_\delta F(k))^{-1} \delta V_\delta \frac{A_{-1}}{k} \text{ with } V_\delta \text{ a differential operator.} \]

- Feschbar-Grushin decomposition: in a suitable basis, this operator writes

\[ I + (I + \delta V_\delta F(k))^{-1} \delta V_\delta \frac{A_{-1}}{k} = \begin{pmatrix} I & * \\ 0 & a \end{pmatrix} \text{ with } a = 1 + \frac{\eta(k)}{k} = \frac{k + \eta(k)}{k} \]

- Here \( \eta \) is an analytical function. The resonances are the \( k \) such that

\[ k + \eta(k) = 0. \]

Conclude with Rouché theorem and asymptotic analysis as \( \delta \rightarrow 0 \).
A general procedure for fibers operators

Consider an analytically fibered operator

- In 1d, any **threshold** given by a **non degenerate unique critical point** is a **meromorphic branching point** of the resolvent.

\[ R_0(k) = \frac{iGG^*}{k} + F(k). \]

- General theory when the band functions are proper in ([Gérard-Nier 98]).

Add a small perturbation \( \delta V_0 \)

- Existence of a **unique resonance** \( k(\delta) \) near each of these thresholds in a general framework ([Grigis-Klopp 95]).

- We have shown the expected formula
  \[ k(\delta) = i\eta_1 \delta + i\eta_2 \delta^2 + O(\delta^2) \text{ with } \eta_1 = GV_0G^*. \]

- The next term \( \eta_2 \) is given by \( \eta_2 = -GV_0F(0)V_0G^*. \)

- For the Laplacian, \( F(0) \) is an explicit convolution operator. In the general case, it can be expressed with Hadamard regularization of Cauchy type integral.
The Neumann magnetic Laplacian

My favorite Hamiltonian:

- The magnetic Laplacian with unitary magnetic field and Neumann boundary condition:

\[ H_0 = (-i \nabla - A)^2 = -\partial_t^2 + (-i \partial_s - t)^2 \quad \text{in} \quad \mathbb{R}^2_+ := \mathbb{R} \times \mathbb{R}_+ \]

- Fibered by partial Fourier transform in \( s \).
  Each band function has a unique non degenerate minimum.
Proofs and generalizations

Small deformation of the boundary

- Parametrize the boundary by a function $\delta \chi$ with
  
  \[ \delta \ll 1 \quad \text{and} \quad \lim_{\pm \infty} \chi = 0. \]

- When $\chi \geq 0$:
  - $\delta > 0$ models an obstacle and $\delta < 0$ a bump.
  - Consider $(-i \nabla - A)^2 = -\partial_t^2 + (-i \partial_s - t)^2$ in $L^2(\Omega)$ with Neumann boundary condition.
Another geometric perturbation

The perturbed operator

- After rectification of the boundary:

\[ H_\delta = -\partial_x^2 + (-i\partial_y - x - i\delta\chi'(y)\partial_x + \delta\chi(y))^2 \text{ in } \mathbb{R}_+^2 \]

- New boundary condition:

\[ \partial_x u = \frac{1}{(1 + \delta^2\chi'(y)^2)} (\delta\chi'(y)\partial_y u - i\delta^2\chi'(y)\chi(y)) \text{ at } x = 0 \]

- Since \( H_\delta \) and \( H_0 \) have different domains, you cannot write \( V = H_\delta - H_0 \).

Approach with a difference of resolvent:

- Consider the difference of resolvent

\[ W_\delta = H_\delta^{-1} - H_0^{-1} \text{ acting on } L^2(\mathbb{R}_+^2). \]

- We found that \( W_\delta = H_0^{-1}V_\delta H_\delta^{-1} \) where

\[ V_\delta = \text{second order differential operator} + \text{boundary operator}. \]

- Use more resolvent identities.
Proofs and generalizations

Resonance or eigenvalue?

Recall that $\sigma(H_0) = [\Theta_0, +\infty)$.

**Theorem (B.G.P. 2020)**

Fix a sufficiently small neighborhood of zero $\mathcal{D}$ in $\mathbb{C}$. There exists $\delta_0 > 0$ such that for $\delta \leq \delta_0$, the function $\mathbb{C}^+ \ni k \mapsto (H_{\delta} - \Theta_0 - k^2)^{-1}$, acting on $wL^2(\mathbb{R}_+^2)$, admits a meromorphic extension on $\mathcal{D}$. This function has a unique pole $k(\delta)$ in $\mathcal{D}$. It has multiplicity one and satisfies

$$k(\delta) = i\mu_2\delta^2 + O(\delta^3), \quad \mu_2 \in \mathbb{R}.$$  

Geometrical comments:

- Changing $\delta$ in $-\delta$ does not change the main asymptotics: a bump or a hole creates the same effect.
- The main term $\mu_2$ depends on $\|\chi\|_{H^1}$ and $\|\hat{\chi}\|_{L^2}$ (upcoming result).
Birman Schwinger form and Grushin matrix:

Define \( H_\delta - H_0 = \delta V_\delta \). It is a second order differential operator.

Write \( R_0(k) = (H_0 - \mathcal{E}_1 - k^2)^{-1} = \frac{A_{-1}}{k} + F(k) \) with \( F \) holomorphic.

Formal resolvent identity for \( R(k) = (H_\delta - \mathcal{E}_1 - k^2)^{-1} \) in weighted space:

\[
R(k) = R_0(k) (I + \delta V_\delta F(k))^{-1} \left( I + (I + \delta V_\delta F(k))^{-1} \delta V_\delta \frac{A_{-1}}{k} \right)^{-1}
\]

Therefore, \( R(k) \) is well defined iff \( I + (I + \delta V_\delta F(k))^{-1} \delta V_\delta \frac{A_{-1}}{k} \) is invertible.

Here, \( A_{-1} \) is rank 1. In a basis adapted to \( \ker(A_{-1}) \oplus \operatorname{Im}(A_{-1}) \), we have:

\[
l + (l + \delta V_\delta F(k))^{-1} \delta V_\delta \frac{A_{-1}}{k} = \begin{pmatrix} 1 & * \\ 0 & a \end{pmatrix}
\]

with \( a = 1 + \frac{\eta(k)}{k} = \frac{k + \eta(k)}{k} \)

Consequence: the resonances are the \( k \) such that \( k + \eta(k) = 0 \).

Write \( A_{-1} = iG^* G \) with \( G \) a linear form, so that

\[
\eta(k) = i\delta G \left( l + \delta V_\delta F(k) \right)^{-1} V G^*
\]