Eigenvalue and resonance asymptotics in perturbed waveguide : twisting versus bending

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Numerical Waves
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Plan

Fibered operators

Diagonalization of an operator:

- Models with invariance properties are reduced to lower dimensional problems.
  Translation invariance : partial Fourier transform :
  $\rightarrow$ Magnetic field, waveguides, stratified media.
  Periodic invariance : Floquet-Bloch transform.
  $\rightarrow$ Cristallin structures, Graphene.
- General framework: an operator H_0 with invariance properties writes

$$UH_0U^* = \int^{\oplus} h_0(p)\mathrm{d}p, \text{ with } U \text{ a unitary transform.}$$

- Fiber operators $h_0(p)$ (may) have a discrete sectrum $(E_n(p))_{n\geq 1}$.

Spectrum of H_0:

- The band functions are $p \mapsto E_n(p)_{n\geq 1}$. The spectrum of H_0 is

$$\sigma(H_0) = \overline{\underset{n\geq 1}{\cup} \mathrm{Ran}\, E_n}.$$

- Non constant band functions correspond to absolutely continuous spectrum.

Some questions (among others)

Transport properties

- Consider the Schrödinger equation

$$i\partial_t \psi = H_0 \psi.$$

Spectral analysis of H_0 requiered for the time dependant analysis.

- Gaps in the spectrum correspond to the absence of propagation in the direction of invariance.
- After perturbation, discrete eigenvalues correspond to trapped modes.
- Resonances play a role in scattering theory.

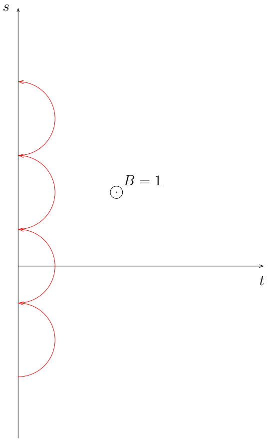
Spectral analysis

- Critical points of the band functions are thresolds in the spectrum of H_0. General theory in [GeNi98] for proper analytical band functions.
- They correspond to instable energies of the system.
  $\rightarrow$ No standard limiting absorption principle.
  $\rightarrow$ More eigenvalues and resonances after perturbation.

Somes examples: The magnetic Laplacian

Constant magnetic field $B=1$ in a half-plane $\mathbb{R}^2_+ = \{(s,t) \in \mathbb{R} \times \mathbb{R}_+\}$.

Classical trajectories:



Somes examples: The magnetic Laplacian

The Schrödinger operator:

- The magnetic potential $A(s,t) = (-t, 0)$ generates a magnetic field

$$B = \mathrm{curl}\ A = 1.$$

- The magnetic Laplacian is (the closure of)

$$H_0 = (-i\nabla - A)^2 = -\partial_t^2 + (-i\partial_s - t)^2 \text{ in } L^2(\mathbb{R}_+^2)$$

  with (let's say) Neumann boundary conditions.

Reduction of dimension:

- Here, U is the partial Fourier transform in s. The fiber operator is for $p \in \mathbb{R}$:

$$h_0(p) = -\partial_t^2 + (t-p)^2 \text{ in } L^2(\mathbb{R}_+).$$

- The band functions are the values E for which there exists $y \neq 0$ solution of

$$\begin{cases} -y''(t) + (t-p)^2 y(t) = Ey(t), & t > 0, \\ y'(0) = 0. \end{cases}$$

Somes examples: The magnetic Laplacian

Each band function E_n has a unique non-degenerate minimum ([Dg,DH93]).

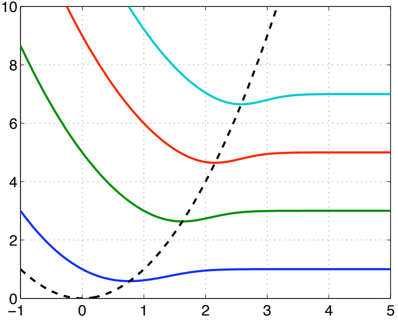
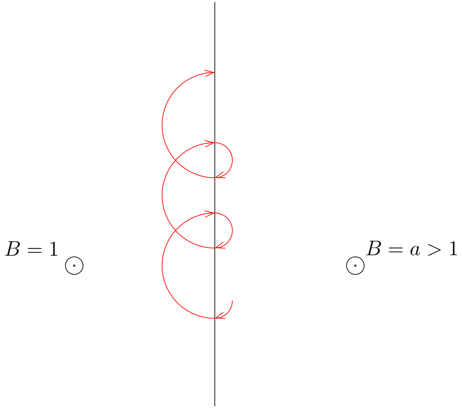


Figure: Abscissa: p (Fourier parameter dual to s).

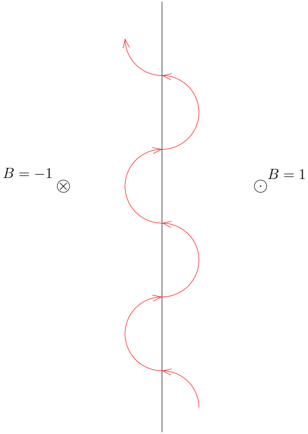
$$\sigma(H_0) = [\min E_1, +\infty).$$

It also tends to a finite limit (Landau levels)!

More magnetic models: the snake's orbits

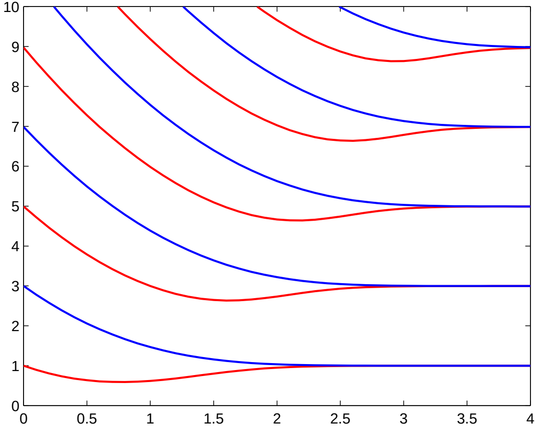


More magnetic models: the snake's orbits



More magnetic models: the snake's orbits

Band functions for the symmetric magnetic step :



Somes examples: The Laplacian in a perturbed waveguide

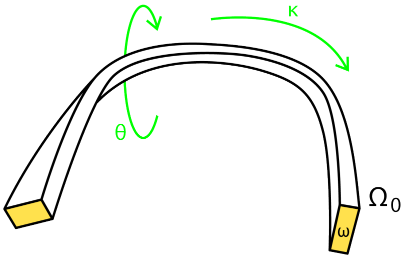
To be continued

Plan

Construction of the waveguide

Geometric data of the waveguide:

- A reference curve $\gamma : \mathbb{R} \to \mathbb{R}^3$, defined by a curvature κ and a torsion.
- A cross section $\omega \subset \mathbb{R}^2$, an open bounded lipschitz domain.
- A rotation $\theta : \mathbb{R} \to \mathbb{R}$ of the section around γ.



To make it simpler, assume that the torsion of γ is 0.
The rotation θ will play a similar role...

Tubular coordinate

Construction of a waveguide $\Omega \subset \mathbb{R}^3$:

- Let (e_1, e_2, e_3) be a Frenet frame associated with γ.
- Let $e_2^\theta(s), e_3^\theta(s)$ be the frame obtained by applying a rotation of angle $\theta(s)$ to $(e_2(s), e_3(s))$ around $e_1(s)$.
- Define $\mathcal{L} : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^3$ by

$$\mathcal{L}(s, t_2, t_3) = \gamma(s) + t_2 e_2^\theta(s) + t_3 e_3^\theta(s)$$

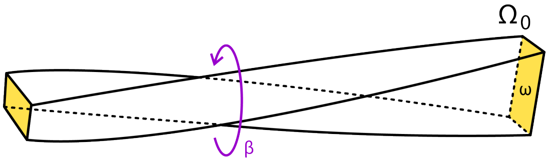
With additional hypotheses, $\mathcal{L}$ is a diffeomorphism on $\mathbb{R} \times \omega$, and the waveguide is defined by

$$\Omega := \mathcal{L}(\mathbb{R} \times \omega).$$

The reference unperturbed operator

The reference tube Ω_0:

- The curve $\gamma(\mathbb{R})$ is a line (i.e. $\kappa = 0$).
- The twisting θ' is a constant β, with two different case:
  $\beta = 0$: straight tube (a cylinder)
  $\beta \neq 0$: periodically twisted tube.



The reference unperturbed operator

Our reference operator:
The Laplacian $-\Delta$ in Ω_0 with Dirichlet boundary conditions.
Denote φ the cylindrical variable and $\partial_\varphi = t_2\partial_3 - t_3\partial_2$ the angular derivative.
After a change of variable the original Laplacian is unitarily equivalent to

$$H_0 = -\partial_2^2 - \partial_3^2 - (\partial_s - \beta\partial_\varphi)^2 \quad \text{in} \quad L^2(\mathbb{R}\times\omega) \quad \text{with Dirichlet b.c.}$$

Let $\mathcal{F}_s$ be the Partial Fourier transform in the s variable.

$$\text{Fiber decomposition:} \quad \mathcal{F}_s H_0 \mathcal{F}_s^* = \int_{p\in\mathbb{R}}^{\oplus} h_0(p)\,\mathrm{d}p.$$

$$h_0(p) := -\Delta_\omega - (-ip - \beta\partial_\varphi)^2 \quad \text{in} \quad L^2(\omega) \quad \text{with Dirichlet b.c.}$$

Band functions of the free operator

The operators $(h_0(p))_{p\in\mathbb{R}}$ form a type A analytic family of self-adjoint operators with compact resolvent.
Denote by $(E_n(p))_{n\geq 0}$ the increasing sequence of eigenvalues of $h_0(p)$. Then

$$\sigma(H_0) = \overline{\cup_{n\geq 1} E_n(\mathbb{R})} = [\mathcal{E}_1, +\infty) \quad \text{with} \quad \mathcal{E}_n = \min_{p\in\mathbb{R}} E_n(p).$$

Analysis of the first band function, done in [Briet-Kovarik-Raikov-Soccorsi 08]:

- The first eigenvalue $E_1(p)$ is non-degenerate (simple).
- $\mathcal{E}_1 = E_1(0)$ and this minimum is non-degenerate and unique.

The case $\beta = 0$ of a straight tube

- In the cylinder $\mathbb{R}\times\omega$, the variables decouple:

$$-\Delta_{\Omega_0} = \mathrm{Id}\otimes(-\Delta_\omega) + (-\partial_s^2)\otimes\mathrm{Id}$$

- The band functions are explicit:

$$E_n(p) = \mathcal{E}_n + p^2$$

here $(\mathcal{E}_n)_{n\geq 1}$ are the eigenvalues of $-\Delta_\omega$ (with Dirichlet b.c.).

Perturbation of the waveguide

Let κ and τ be two real functions such that

$$\lim_{\pm\infty} \kappa = \lim_{\pm\infty} \tau = 0.$$

Consider $\Omega = \mathcal{L}(\mathbb{R} \times \omega)$, the tube of cross section ω, along the curve γ and twisted by θ with $\theta' = \beta + \tau$.
Let H be the Dirichlet Laplacian in Ω. The metric is $G := (\mathrm{d}\mathcal{L})^T (\mathrm{d}\mathcal{L})$.

$$G = \begin{pmatrix} h^2 + h_2^2 + h_3^2 & h_2 & h_3 \\ h_2 & 1 & 0 \\ h_3 & 0 & 1 \end{pmatrix} \text{ with } \begin{cases} h(s,t) = 1 - \kappa(s)(t_2 \cos\theta(s) + t_3 \sin\theta(s)) \\ h_2(s,t) = t_3 \theta'(s) \\ h_3(s,t) = -t_2 \theta'(s) \end{cases}$$

Remark that $h = \sqrt{\det(G)}$, and note $G^{-1} = G^{jk}$. The operator is

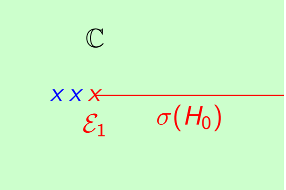
$$-\Delta_\Omega \equiv \frac{1}{h} \sum_{j,k=1}^{3} \partial_j h G^{jk} \partial_k \quad \text{on} \quad L^2(\mathbb{R} \times \omega, h) \quad \text{with Dirichlet b.c..}$$

Creation of eigenvalues

After a change of variables:

$$H \equiv -\partial_2^2 - \partial_3^2 - \frac{\kappa^2}{4h^2} - \left(h^{-1/2}(\partial_s - \theta' \partial_\varphi)h^{-1/2}\right)^2 \text{ in } L^2(\mathbb{R}\times\omega) \text{ with Dirichlet b.c.}$$

Its essential spectrum is still $[\mathcal{E}_1, +\infty)$. What about discrete spectrum?



Twisting vs bendig: references

(Some) Known results for $\beta = 0$

- Pure bending ($\theta' = 0$ and $\kappa \neq 0$) creates discrete eigenvalues (Duclos-Exner 95, Grushin 04).
- Pure twisting ($\kappa = 0$ and $\tau \neq 0$) does not change the spectrum (Grushin 04). Existence of a Hardy inequality (Ekholm-Kovarik-Krejcirik 08) proves that adding a small bending ($\kappa \ll 1$) does not add discrete spectrum.

(Some) Known results for $\beta > 0$ and $\kappa = 0$.

- Small enhanced twisting $0 < \tau \ll 1$ does not change the spectrum ([Briet-Hammadi-Krejkirik 15])
- Slowed twisting ($\int \tau < 0$) creates eigenvalues (eventually an infinite number). Counting function studied in Briet-Kovarik-Raikov-Soccorsi 08].
- Scattering properties in Briet-Kovarik-Raikov 15].

Our perturbative approach

Some questions:

- No result when $\beta \neq 0$ and $\kappa \neq 0$.
- For straight tubes, switching from bending to twisting, eigenvalue(s) diseappear in the essential spectrum. Are they resonances?
- Provide quantitative criterion to compare twisting and bending, even when $\beta \neq 0$.

Our approach:

- Consider a waveguide along a curve with curvature $\delta\kappa$, with cross section ω fixed, and twisted by a rotating function $\theta' = \beta + \delta\tau$.
- Study what happen near $\mathcal{E}_1$ as $\delta \to 0$

Recall that the reference operator is

$$H_0 = -\partial_2^2 - \partial_3^2 - (\partial_s - \beta\partial_\varphi)^2.$$

The perturbed operator is

$$H_\delta := -\partial_2^2 - \partial_3^2 - \delta^2 \frac{\kappa^2}{4h_\delta^2} - (h_\delta^{-1/2}(\partial_s - (\beta+\delta\tau)\partial_\varphi)h_\delta^{-1/2})^2$$

with $h_\delta(s,t) = 1 - \delta\kappa(s)(t_2 \cos\theta_\delta(s) + t_3 \sin\theta_\delta(s))$, $\theta'_\delta = \beta + \delta\tau$.

Plan

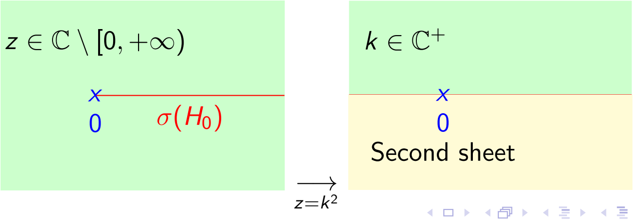
Recall the meromorphic extension for the 1d Laplacian

- $z \mapsto (-\Delta_{\mathbb{R}} - z)^{-1}$ is well defined on $\mathbb{C} \setminus [0, +\infty)$.
- Set $z = k^2$, with $k \in \mathbb{C}^+ = \{im(k) > 0\}$. Then $(-\Delta - k^2)^{-1}$ has a kernel

$$\forall k \in \mathbb{C}^+, \quad \int_{\mathbb{R}} \frac{e^{ip(s-s')}}{p^2 - k^2} \mathrm{d}p = \frac{e^{ik|s-s'|}}{2ik}.$$

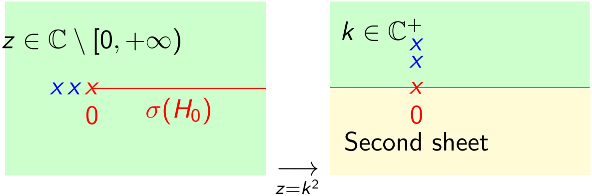
Theorem

Let $w(s) = exp(-a|s|)$ with $a > 0$. Then $k \mapsto (-\Delta_{\mathbb{R}} - k^2)^{-1}$, acting on $wL^2(\mathbb{R})$, initially defined on $\mathbb{C}^+$, admits a meromorphic extension to $\{im(k) > -a\}$, with a unique pole in 0.



Resonances for perturbations

- For suitable V, the resolvent $(-\Delta_{\mathbb{R}} + V - k^2)^{-1}$ is meromorphic on a neighborhood of 0.
- Its poles are the resonances of $-\Delta_{\mathbb{R}} + V$.
  For small V, you expect a single resonance near 0.
- When $k \in \mathbb{C}^+ \cap i\mathbb{R}$ is a resonance, it corresponds to a negative eigevalue.



Motivations for finding resonances

- Expansion of the semi-group: near an isolated resonance k_0 for H,

$$e^{-itH} \approx e^{-itz_0}\Pi + R \quad \text{with} \quad \begin{cases} \Pi \text{ rank one projector} \\ R \text{ remainder} \end{cases}$$

- Singularities of the spectral shift function and Breit-Wigner formulas.
- Poles of the scattering matrix

Resolvent of our free opetator

Lemma

The free resolvent $\mathbb{C}^+ \ni k \mapsto (H_0 - \mathcal{E}_1 - k^2)^{-1}$, acting on the weighted space $wL^2(\mathbb{R} \times \omega)$, admits a meromorphic extension near 0.

- Proof: extend singular Cauchy integrals of the form

$$\int_{\mathbb{R}} e^{ip(s-s')}\psi_n(t,p)\psi_n(t',p)(E_n(p) - \mathcal{E}_1 - k^2))^{-1}\mathrm{d}p,$$

- Similar results near other non degenerate critical points of the band functions, in particular near each μ_n when $\beta = 0$.
- General method for analitically fibered operators, see [Gérard 90].

Hypotheses on the waveguide:

- The functions κ and τ are C^2.
- These functions, their first and second derivative satisfy

$$\kappa(s), \tau(s) = O(e^{-\alpha s^2})$$

for some $\alpha > 0$.

Main result for periodically twisted waveguide

Theorem (B.M.P.P 2018)

Fix a sufficiently small neighborhood of zero $\mathcal{D}$ in $\mathbb{C}$. Then, there exists $\delta_0 > 0$ such that for $\delta \leq \delta_0$, the function $\mathbb{C}^+ \ni k \mapsto (H_\delta - \mathcal{E}_1 - k^2)^{-1}$, acting on $wL^2(\mathbb{R} \times \omega)$, admits a meromorphic extension on $\mathcal{D}$. This function has a unique pole $k(\delta)$ in $\mathcal{D}$. It has multiplicity one and satisfies

$$k(\delta) = i\mu_1\delta + O(\delta^2), \quad \mu_1 \in \mathbb{R}.$$

Moreover, there exists a function $F : \mathbb{R} \to \mathbb{R}$, constants $c_j > 0$, such that

$$\mu_1 = -c_1\beta \int_{\mathbb{R}} \tau(s)ds + c_2\beta^2 \int_{\mathbb{R}} \kappa(s)F(s)ds.$$

The function F, c_1 and c_2 depend only on β and ω (explicit).
Further, the pole $k(\delta)$ is a purely imaginary number.

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The function F, c_1 and c_2 depend only on β and ω (explicit).
Further, the pole $k(\delta)$ is a purely imaginary number.

$$F(s) = \int_\omega \left(|\partial_\varphi\psi_1(t)|^2 + \frac{1}{4}|\psi_1(t)|^2\right)(t_2\cos(\beta s) + t_3\sin(\beta s))\mathrm{d}t,$$

Comments on the result

Recall that our resonance of $H_\delta - \mathcal{E}_1$ satisfies

$$k(\delta) = i\mu_1\delta + O(\delta^2), \quad \mu_1 \in \mathbb{R}.$$

with

$$\mu_1 = -c_1\beta \int_{\mathbb{R}} \tau(s)ds + c_2\beta^2 \int_{\mathbb{R}} \kappa(s)F(s)ds.$$

Localization of the resonance:

- When $\mu_1 > 0$,

$$\mathcal{E}_1 + k(\delta)^2 = \mathcal{E}_1 - \delta^2\mu_1^2 + O(\delta^3) \text{ is a discrete eigenvalue below } \mathcal{E}_1.$$

- When $\mu_1 < 0$, it gives an antiboundstate.
- When $\mu_1 = 0$, you need to go to the next order (hard).

Influence of the geometry:

- When $\kappa = 0$, it depends on the sign of $\int_{\mathbb{R}} \tau$.
- When $\int_{\mathbb{R}} \tau = 0$, both cases can appear.
- When $\beta \ll 1$, you can focus on the first term.
- When $\beta = 0$, you need to go to the next order.

Main result for perturbation of a straight waveguide

Theorem (B.M.P.P 2018)

Assume $\beta = 0$, the same results hold, but $k(\delta)$ satisfies

$$k(\delta) = i\mu_2\delta^2 + O(\delta^3), \quad \mu_2 \in \mathbb{R}.$$

Moreover, there exist positive bilinear form q_1, q_2, and a bilinear form q_3 such that

$$\mu_2 = q_1(\kappa, \kappa) - q_2(\tau, \tau) + q_3(\tau, \dot{\kappa})$$

Similar result near each threshold $\mathcal{E}_n$, depending on its multiplicity as an eigenvalues of $-\Delta_\omega$.

Main result for perturbation of a straight waveguide

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$$\mu_2 = q_1(\kappa, \kappa) - q_2(\tau, \tau) + q_3(\tau, \dot{\kappa})$$

$$\begin{aligned}\mu_2 =& \frac{1}{8}\sum_{q\geq 2}(\mathcal{E}_q - \mathcal{E}_1)^2 \langle \psi_q | t_2\psi_1\rangle^2 \langle \kappa | (-\partial_s^2 + \mathcal{E}_q - \mathcal{E}_1)^{-1}\kappa\rangle \\ &- \frac{1}{2}\sum_{q\geq 2}(\mathcal{E}_q - \mathcal{E}_1)\langle \psi_q | \partial_\varphi \psi_1\rangle^2 \langle \tau | (-\partial_s^2 + \mathcal{E}_q - \mathcal{E}_1)^{-1}\tau\rangle \\ &+ \frac{1}{2}\sum_{q\geq 2}(\mathcal{E}_q - \mathcal{E}_1)\langle \psi_q | \partial_\varphi \psi_1\rangle \langle \psi_q | t_2\psi_1\rangle \langle \tau | (-\partial_s^2 + \mathcal{E}_q - \mathcal{E}_1)^{-1}\dot{\kappa}\rangle.\end{aligned}$$

Plan

Birman Schwinger form and Grushin matrix:

Main ingredients:

- Laurent expansion of the free resolvent:

$$R_0(k) = (H_0 - \mathcal{E}_1 - k^2)^{-1} = \frac{A_{-1}}{k} + F(k) \quad \text{with} \quad F \quad \text{holomorphic near } 0$$

- Resolvent identity and Birman Schwinger principle led to invert

$$I + (I + \delta V_\delta F(k))^{-1} \delta V_\delta \frac{A_{-1}}{k} \quad \text{with } V_\delta \text{ a differential operator.}$$

- Feschbar-Grushin decomposition: in a suitable basis, this operator writes

$$I + (I + \delta V_\delta F(k))^{-1} \delta V_\delta \frac{A_{-1}}{k} = \begin{pmatrix} I & \star \\ 0 & a \end{pmatrix} \quad \text{with} \quad a = 1 + \frac{\eta(k)}{k} = \frac{k + \eta(k)}{k}$$

- Here η is an analytical function. The resonances are the k such that

$$k + \eta(k) = 0.$$

Conclude with Rouché theorem and asymptotic analysis as $\delta \to 0$.

A general procedure for fibers operators

Consider an anlytically fibered operator

- In 1d, any threshold given by a non degenerate unique critical point is a meromorphic branching point of the resolvent.

$$R_0(k) = \frac{iGG^*}{k} + F(k).$$

- General theory when the band functions are proper in ([Gérard-Nier 98]).

Add a small perturbation δV_0

- Existence of a unique resonance $k(\delta)$ near each of these thresholds in a general framework ([Grigis-Klopp 95]).
- We have shown the expected formula

$$k(\delta) = i\eta_1\delta + i\eta_2\delta^2 + O(\delta^2) \quad \text{with} \quad \eta_1 = GV_0G^*.$$

- The next term η_2 is given by $\eta_2 = -GV_0F(0)V_0G^*$.
- For the Laplacian, $F(0)$ is an explicit convolution operator.
  In the general case, it can be expressed with Hadamard regularization of Cauchy type integral.

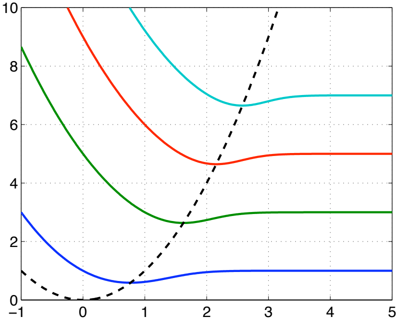
The Neumann magnetic Laplacian

My favorite Hamiltonian:

- The magnetic Laplacian with unitary magnetic field and Neumann boundary condition:

$$H_0 = (-i\nabla - A)^2 = -\partial_t^2 + (-i\partial_s - t)^2 \text{ in } \mathbb{R}^2_+ := \mathbb{R} \times \mathbb{R}_+$$

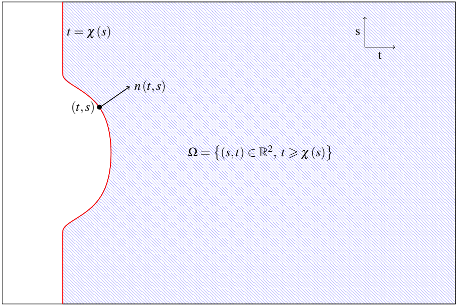
- Fibered by partial Fourier transform in s.
  Each band function has a unique non degenerate minimum.



Small deformation of the boundary

- Parametrize the boundary by a function $\delta\chi$ with

$$\delta \ll 1 \quad \text{and} \quad \lim_{\pm\infty} \chi = 0.$$



- When $\chi \geq 0$:
  $\delta > 0$ modelizes an obstacle and $\delta < 0$ a bump.
- Consider $(-i\nabla - A)^2 = -\partial_t^2 + (-i\partial_s - t)^2$ in $L^2(\Omega)$ with Neumann boundary condition.

Another geometric perturbation

The perturbed operator

- After rectification of the boundary:

$$H_\delta = -\partial_x^2 + (-i\partial_y - x - i\delta\chi'(y)\partial_x + \delta\chi(y))^2 \quad \text{in} \quad \mathbb{R}^2_+$$

- New boundary condition :

$$\partial_x u = \frac{1}{(1+\delta^2\chi'(y)^2)}\left(\delta\chi'(y)\partial_y u - i\delta^2\chi'(y)\chi(y)\right) \quad \text{at} \quad x = 0$$

- Since H_δ and H_0 have different domains, you cannot write $V = H_\delta - H_0$.

Approach with a difference of resolvent:

- Consider the difference of resolvent

$$W_\delta = H_\delta^{-1} - H_0^{-1} \quad \text{acting on} \quad L^2(\mathbb{R}^2_+).$$

- We found that $W_\delta = H_0^{-1} V_\delta H_\delta^{-1}$ where

$$V_\delta = \text{second order diferential operator} + \text{boundary operator}.$$

- Use more resolvent identities.

Resonance or eigenvalue?

Recall that $\sigma(H_0) = [\Theta_0, +\infty)$.

Theorem (B.G.P. 2020)

Fix a sufficiently small neighborhood of zero $\mathcal{D}$ in $\mathbb{C}$. There exists $\delta_0 > 0$ such that for $\delta \leq \delta_0$, the function $\mathbb{C}^+ \ni k \mapsto (H_\delta - \Theta_0 - k^2)^{-1}$, acting on $wL^2(\mathbb{R}^2_+)$, admits a meromorphic extension on $\mathcal{D}$. This function has a unique pole $k(\delta)$ in $\mathcal{D}$. It has multiplicity one and satisfies

$$k(\delta) = i\mu_2\delta^2 + O(\delta^3), \quad \mu_2 \in \mathbb{R}.$$

Geometrical comments:

- Changing δ in $-\delta$ does not change the main asymptotics: a bump or a hole creates the same effect.
- The main term μ_2 depends on $\|\chi\|_{H^1}$ and $\|\hat{\chi}\|_{L^2}$ (upcoming result).

Birman Schwinger form and Grushin matrix:

Define $H_\delta - H_0 = \delta V_\delta$. It is a second order differential operator.
Write $R_0(k) = (H_0 - \mathcal{E}_1 - k^2)^{-1} = \frac{A_{-1}}{k} + F(k)$ with F holomorphic.
Formal resolvent identity for $R(k) = (H_\delta - \mathcal{E}_1 - k^2)^{-1}$ in weighted space:

$$R(k) = R_0(k)\left(I + \delta V_\delta F(k)\right)^{-1}\left(I + \left(I + \delta V_\delta F(k)\right)^{-1}\delta V_\delta \frac{A_{-1}}{k}\right)^{-1}$$

Therefore, $R(k)$ is well defined iff $I + \left(I + \delta V_\delta F(k)\right)^{-1}\delta V_\delta \frac{A_{-1}}{k}$ is invertible.
Here, A_{-1} is rank 1. In a basis adapted to $\ker(A_{-1}) \bigoplus \mathrm{Im}(A_{-1})$, we have:

$$I + \left(I + \delta V_\delta F(k)\right)^{-1}\delta V_\delta \frac{A_{-1}}{k} = \begin{pmatrix} I & \star \\ 0 & a \end{pmatrix} \quad \text{with} \quad a = 1 + \frac{\eta(k)}{k} = \frac{k + \eta(k)}{k}$$

Consequence: the resonances are the k such that

$$k + \eta(k) = 0.$$

Write $A_{-1} = iG^* G$ with G a linear form, so that

$$\eta(k) = i\delta G\left(I + \delta V_\delta F(k)\right)^{-1} V G^*$$