A non-linear variational characterization for Dirac eigenvalues in bounded domains.

Application to a spectral geometric inequalities.

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Joint work with

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Numerical waves
Nice
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Setting of the problem

Geometrical setting : $\Omega \subset \mathbb{R}^2$ such that

- Ω is C^∞, bounded, simply connected;
- $\nu = (\nu_1, \nu_2)$ is the outward pointing normal vector field (we set $\mathbf{n} = \nu_1 + i\nu_2$).

(Euclidean) Dirac operator : Operator which acts in $L^2(\Omega, \mathbb{C}^2)$ and acts as

$$D^\Omega := -i\sigma_1\partial_1 - i\sigma_2\partial_2 = \begin{pmatrix} 0 & -2i\partial_z \\ -2i\partial_{\bar{z}} & 0 \end{pmatrix},$$

where the Pauli matrices $\sigma_1, \sigma_2, \sigma_3$ are

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We set

$$\mathrm{dom}(D^\Omega) := \{u = (u_1, u_2)^\top \in H^1(\Omega, \mathbb{C}^2) : u_2 = i\mathbf{n}u_1 \text{ on } \partial\Omega\}.$$

The spectral problem: find $(E, u) \in \mathbb{R} \times \mathrm{dom}(D^\Omega)$ such that

$$\boxed{D^\Omega u = Eu.}$$

From where does this operator comes from ?

- Introduced by Dirac in 1928 to have a quantum theory taking into account the spin & special relativity.
- With this boundary condition it appears as a shape optimization problem introduced by particle physicists to model the confinement in hadrons:

$$\inf_{\Omega}\{E_1(\Omega) + b|\Omega|\}, \quad b > 0$$

where $E_1(\Omega) > 0$ is the first non-negative eigenvalue of D^{Ω}.

P. N. BOGOLIUBOV
ANN. INST. HENRI POINCARÉ (1968)

A. CHODOS, R. L. JAFFE, K. JOHNSON, C. B.THORN
PHYS. REVIEW D (1974)

- Renewal of interest : 2D operator is an effective operator to describe the behavior of electrons in graphene nanostructure.

About the boundary condition

The boundary condition is obtained as the limit when $m \to +\infty$ of the operator posed in $L^2(\mathbb{R}^2, \mathbb{C}^2)$:

$$" \quad -i\sigma_1\partial_1 - i\sigma_2\partial_2 + m\mathbb{1}_{\Omega^c}\sigma_3 \underset{m\to+\infty}{\longrightarrow} D^\Omega \quad "$$

J. M. Barbaroux, H. Cornean, L. Le Treust, E. Stockmayer
Annales Henri Poincaré (2019)

N. Arrizabalaga, L. Le Treust, A. Mas, N. Raymond
Journal de l'École Polytechnique (2019)

A. Moroianu, T. O.-B., K. Pankrashkin
Communications in Mathematical Physics (2020)

Remark : For $-\Delta^\Omega$ (the Dirichlet Laplacian posed in a domain Ω) can be seen as the limit as $R \to +\infty$

$$" \quad -\Delta + R\mathbb{1}_{\Omega^c} \underset{R\to+\infty}{\longrightarrow} -\Delta^\Omega \quad "$$

Statement of the problem

Proposition

$(D^{\Omega}, \mathrm{dom}(D^{\Omega}))$ is self-adjoint, its spectrum is discrete and verifies

$$\cdots \leq -E_2(\Omega) \leq -E_1(\Omega) < 0 < E_1(\Omega) \leq E_2(\Omega) \leq \cdots$$

R. D. Benguria, S. Fournais, E. Stockmeyer, H. Van Den Bosch
Annales Henri Poincaré. (2017)

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R. D. Benguria, S. Fournais, E. Stockmeyer, H. Van Den Bosch
Annales Henri Poincaré. (2017)

They were interested in proving a Faber-Krahn inequality for $E_1(\Omega)$.

Theorem (in dimension two)

Let $\Omega \subset \mathbb{R}^2$ be a Lipschitz bounded domain and let $\lambda_1^{\mathrm{Dir}}(\Omega)$ be its first Dirichlet eigenvalue. There holds
$$\frac{\pi}{|\Omega|}\lambda_1^{\mathrm{Dir}}(\mathbb{D}) \leq \lambda_1^{\mathrm{Dir}}(\Omega)$$
with equality if and only if Ω is a disk.

G. Faber
Münch. Ber. (1923)

E. Krahn
Math. Ann. (1925)

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R. D. Benguria, S. Fournais, E. Stockmeyer, H. Van Den Bosch
Annales Henri Poincaré. (2017)

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Faber-Krahn conjecture for Dirac

There holds

$$\sqrt{\frac{\pi}{|\Omega|}} E_1(\mathbb{D}) \le E_1(\Omega)$$

with equality if and only if Ω is a disk.

But in

R. D. Benguria, S. Fournais, E. Stockmeyer, H. Van Den Bosch
Mathematical Physics Analysis and Geometry. (2017)

they prove....

Proposition

$$\sqrt{\frac{2\pi}{|\Omega|}} \leq E_1(\Omega).$$

Some remarks :

- $E_1(\mathbb{D}) \simeq 1.435\ldots$ thus $E_1(\mathbb{D}) > \sqrt{2}$.
- A mode corresponding to $E_1(\mathbb{D})$ is given by

$$u(x) := \begin{pmatrix} J_0(E_1(\mathbb{D})|x|) \\ i\frac{x_1+ix_2}{|x|}J_1(E_1(\mathbb{D})|x|) \end{pmatrix}$$

- This bound was known from people working in spin geometry

S. Raulot
Journal of Geometry and Physics. (2006)

We proved a ”simpler” result :

Theorem [Antunes, Benguria, Lotoreichik, O.-B.]

$$E_1(\Omega) \leq \frac{|\partial\Omega|}{\pi r_i^2 + |\Omega|} E_1(\mathbb{D}),$$

where r_i is the inradius of Ω. There is equality in the previous inequality if and only if Ω is a disk.

A non-linear variational characterization of $E_1(\Omega)$

Goal : obtain a variational characterization of $E_1(\Omega)$.

How ? We develop a min-max principle for our operator (which can be seen as an operator with gap).

J. Dolbeault, M. J. Esteban, E. Séré
J. Funct. Anal. (2000)

M. Griesemer, H. Siedentop
J. London Math. Soc.. (1999)

L. Schimmer, J. P. Solovej, S. Tokus
Ann. Henri Poincaré. (2019)

J. D. Talman
Phys. Rev. Lett. (1986)

To our knowledge : first time this idea is extended to Dirac operators on bounded domain with local boundary conditions.

J.- M. Barbaroux, L. Le Treust, N. Raymond, E. Stockmeyer
preprint. (2020)

Heuristic

Let $E > 0$ and look for $u = (u_1, u_2)^\top \in \mathrm{dom}(D^\Omega)$ such that

$$D^\Omega u = Eu \Longleftrightarrow \left\{ \begin{array}{rcl} -2i\partial_z u_2 & = & Eu_1 \\ -2i\partial_{\bar{z}} u_1 & = & Eu_2 \end{array} \right. \quad \text{in } \Omega.$$

It implies

$$-4\partial_z\partial_{\bar{z}} u_1 = E^2 u_1 \quad \text{in } \Omega. \tag{1}$$

But if $u_2 = -i\frac{2}{E}\partial_{\bar{z}} u_1$ up to the boundary $\partial\Omega$:

$$\bar{\mathbf{n}}\partial_{\bar{z}} u_1 + \frac{E}{2} u_1 = 0 \quad \text{on } \partial\Omega. \tag{2}$$

Multiply (1) by $\bar{u}_1$, integrate by parts (taking into account the b.c. (2)) :

$$4\|\partial_{\bar{z}} u_1\|_\Omega^2 - E^2\|u_1\|_\Omega^2 + E\int_{\partial\Omega} |u_1|^2 ds = 0.$$

Beyond the heurisic

Define for $E > 0$ the quadratic form

$$\begin{cases} q_{E,0}^{\Omega}(u) & := & 4\|\partial_{\bar z} u\|_{\Omega}^2 - E^2\|u\|_{\Omega}^2 + E\int_{\partial\Omega}|u|^2 ds \\ \mathrm{dom}(q_{E,0}^{\Omega}) & = & C^{\infty}(\overline{\Omega}) \end{cases}$$

Proposition

$q_{E,0}^{\Omega}$ is closable and its closure is denoted q_E^{Ω}. Moreover

$$\mathrm{dom}(q_E^{\Omega}) = H^1(\Omega) + \mathcal{H}^2(\Omega).$$

Here :

- $H^1(\Omega)$ is the usual first-order Sobolev space,
- $\mathcal{H}^2(\Omega)$ is the Hardy space (holomorphic functions in Ω with traces $L^2(\partial\Omega)$).

The variational principle

Let $E > 0$:

- q_E^Ω is a densely defined, semi-bounded and closed quadratic form thus associated with a s.a. operator H_E^Ω.
- H_E^Ω has compact resolvent and its first eigenvalue $\mu^\Omega(E)$ is characterized by

$$\mu^\Omega(E) = \inf_{u \in \mathrm{dom}(q_E^\Omega) \setminus \{0\}} \frac{q_E^\Omega(u)}{\|u\|_\Omega^2}$$

Proposition

$\mu^\Omega(E) = 0$ if and only if $E = E_1(\Omega)$.

Elements of proof :

- Study $E \mapsto \mu^\Omega(E)$: continuous, concave, $\mu^\Omega(0) = 0$ (holomorphic functions, loss of compactness), there exists $E_\sharp > 0$ such that for all $E \in (0, E_\sharp)$ there holds$\mu^\Omega(E) > 0$.
- Let $E_\star > 0$ such that $\mu^\Omega(E_\star) = 0$ then for all $E \in (0, E_\star)$ there holds $\mu^\Omega(E) > 0$ and for all $E \in (E_\star, +\infty)$ there holds $\mu^\Omega(E) < 0$.

The variational principle

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Proposition

$\mu^\Omega(E) = 0$ if and only if $E = E_1(\Omega)$.

Elements of proof :

- If $E \in Sp(D^\Omega)$ then $\mu^\Omega(E) \leq 0$.
- If $\mu^\Omega(E) = 0$ then $E \in Sp(D^\Omega)$ (use the regularity of functions in $\mathrm{dom}(H_E^\Omega)$).

Three domains

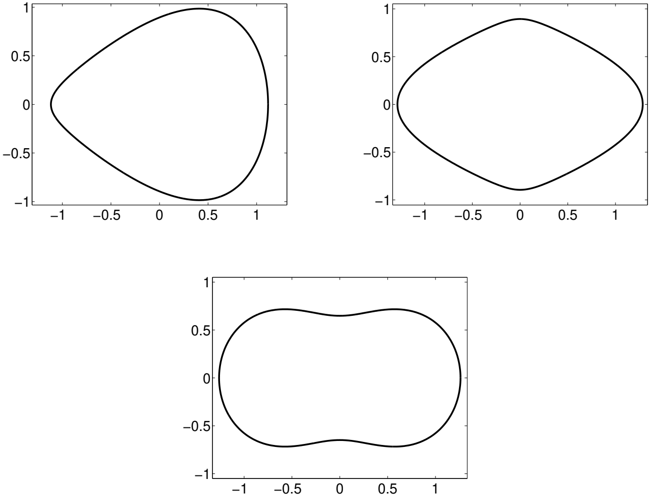


Figure: Three domains of same area of the unit disk denoted $\Omega_1, \Omega_2, \Omega_3$.

Behavior of μ^{Ω}

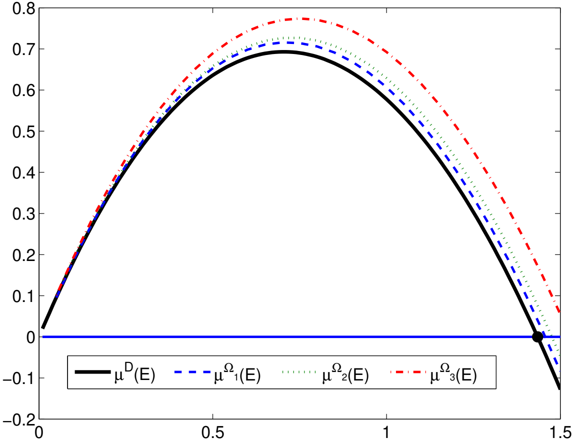


Figure:

Behavior of μ^{Ω_j} with respect to E. The black dot is the first non-negative root $J_0(\lambda) = J_1(\lambda)$ ($\lambda = E_1(\mathbb{D}) \simeq 1.43469565\ldots$).

Upper bounds : philosophy & first result

Thanks to the variational principle, it is easy to obtain upper-bounds : pick the "good" test function such that

$$\frac{q_E^\Omega(u)}{\|u\|_\Omega^2} \leq 0$$

and then $\mu^\Omega(E) \leq 0$ and $E > E_1(\Omega)$.

Proposition (a simple upper bound)

There holds

$$E_1(\Omega) \leq \frac{|\partial\Omega|}{|\Omega|}.$$

<u>Proof</u> : Pick $u \equiv 1$ in Ω, there holds

$$\frac{q_E^\Omega(u)}{\|u\|_\Omega^2} = E\Big(\frac{|\partial\Omega|}{|\Omega|} - E\Big) \quad \text{then chose} \quad E = \frac{|\partial\Omega|}{|\Omega|}.$$

We want to obtain the following upper-bound :

Theorem

There holds

$$E_1(\Omega) \le \frac{|\partial\Omega|}{\pi r_i^2 + |\Omega|} E_1(\mathbb{D}),$$

with equality if and only if Ω is a disk.

This is a consequence of the following theorem

Theorem

There holds

$$E_1(\Omega) \le \frac{|\partial\Omega| + \sqrt{|\partial\Omega|^2 + 8\pi E_1(\mathbb{D})(E_1(\mathbb{D}) - 1)(\pi r_i^2 + |\Omega|)}}{2(\pi r_i^2 + |\Omega|)},$$

with equality if and only if Ω is a disk.

Proof : $\pi r_i^2 \le |\Omega|$, $4\pi|\Omega| \le |\partial\Omega|^2$ and :

$$\begin{aligned} |\partial\Omega|^2 + 8\pi E_1(\mathbb{D})(E_1(\mathbb{D}) - 1)(\pi r_i^2 + |\Omega|) &\le |\partial\Omega|^2(1 + 4E_1(\mathbb{D})(E_1(\mathbb{D}) - 1)) \\ &= |\partial\Omega|^2(2E_1(\mathbb{D}) - 1)^2. \end{aligned}$$

Proof of the upper-bound 1/2

Follow the strategy of Szegö for the first non-trivial e.v. of the Neumann Laplacian :

$$\lambda_1^{\mathrm{Neu}}(\Omega) \leq \frac{\pi}{|\Omega|}\lambda_1^{\mathrm{Neu}}(\mathbb{D}).$$

G. Szegö
J. Ration. Mech. Anal. (1954)

Main idea :

- Without loss of generality : $0 \in \Omega$ and $r_i = \min_{x\in\partial\Omega} \|x\|_{\mathbb{R}^2}$.
- $f : \mathbb{D} \to \Omega$ conformal map, $f(0) = 0$ and $f(z) = \sum_{n=1}^{+\infty} c_n z^n$.
- Consider the minimizer $u_0(x) := J_0(E_1(\mathbb{D})|x|)$ for $\mu^{\mathbb{D}}(E_1(\mathbb{D})) = 0$.
- Use $u := u_0 \circ f^{-1} \in \mathrm{dom}(q_E^{\Omega})$ as a test function.
- Remark that a change of variable gives for all $E > 0$:

$$\mu^{\Omega}(E) \leq \frac{q_E^{\Omega}(u)}{\|u\|_{\Omega}^2} = -E^2 + \frac{4\|\partial_{\bar{z}} u_0\|_{\mathbb{D}}^2 + EJ_0(E_1(\mathbb{D}))^2 \int_0^{2\pi} |f'(e^{i\theta})|d\theta}{\int_{\mathbb{D}} |u_0(x)|^2 |f'(x_1 + ix_2)|^2 dx_1 dx_2}$$

- $\int_0^{2\pi} |f'(e^{i\theta})|d\theta = |\partial\Omega|$.

Proof of the upper bound 2/2

- u_0 is radial so in polar coordinates (using Fourier series) :

$$\int_{\mathbb{D}} |u_0(x)|^2 |f'(x_1 + ix_2)|^2 dx_1 dx_2 = 2\pi \sum_{n=1}^{+\infty} n |c_n|^2 M_n.$$

- Properties of Bessel functions and $E_1(\mathbb{D})$ give:

$$M_n := \int_0^1 J_0(E_1(\mathbb{D})r)^2 r^{2n-1} dr \geq \frac{n}{2n-1} M_1 = \frac{n}{2n-1} J_0(E_1(\mathbb{D}))^2.$$

- Hence

$$2\pi \sum_{n=1}^{+\infty} n|c_n|^2 M_n \geq J_0(E_1(\mathbb{D}))^2 \left(2\pi |c_1|^2 + 2\pi \sum_{n=2}^{+\infty} \frac{n^2}{2n-1} |c_n|^2\right)$$
$$\geq J_0(E_1(\mathbb{D}))^2 \left(\pi r_i^2 + |\Omega|\right)$$

- We get

$$\mu^{\Omega}(E) \leq \frac{P(E)}{\pi r_i^2 + |\Omega|},$$

with $P(E) = -E^2(\pi r_i^2 + |\Omega|) + E|\partial\Omega| + 2\pi E_1(\mathbb{D})(E_1(\mathbb{D}) - 1)$
- Find the roots of $P(E) = 0$.

Some numerics

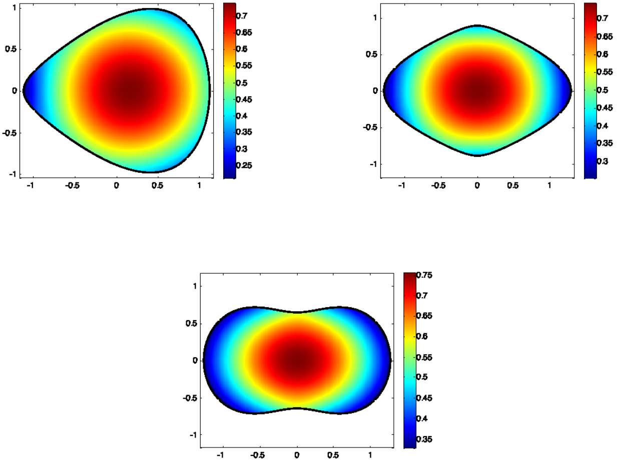


Figure: Modulus of an eigenfunction associated with the first Dirac eigenvalue for $\Omega_1, \Omega_2, \Omega_3$.

Some numerics

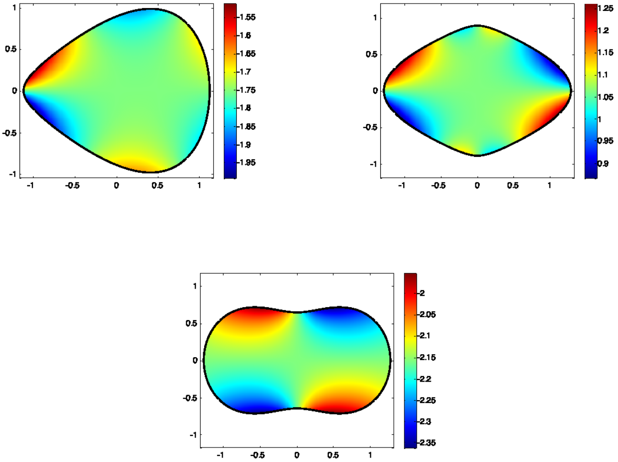


Figure: Argument of an eigenfunction associated with the first Dirac eigenvalue for $\Omega_1, \Omega_2, \Omega_3$.

Back to fhe Faber-Krahn Conjecture

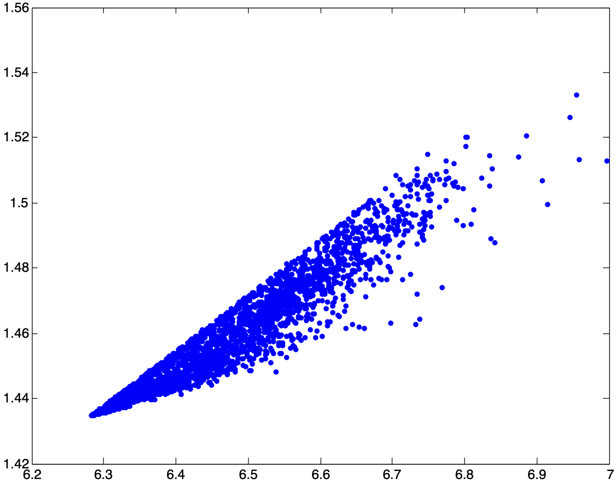
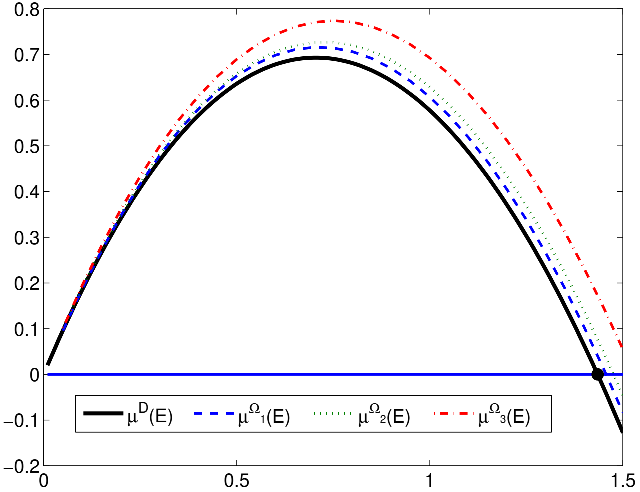


Figure: Plot of the principal eigenvalue for 2500 domains randomly generated satisfying $|\Omega| = \pi$, as function of the perimeter.

Back to fhe Faber-Krahn Conjecture



Behavior of μ^{Ω_j} with respect to E. The black dot is the first non-negative root of $J_0(\lambda) = J_1(\lambda)$ ($\lambda = E_1(\mathbb{D}) \simeq 1.43469565\ldots$).

New conjecture

Faber-Krahn for H^{Ω}_{E}

For all $E > 0$ there holds

$$\frac{\pi}{|\Omega|}\mu^{\mathbb{D}}\Big(\sqrt{\frac{|\Omega|}{\pi}}E\Big) \leq \mu^{\Omega}(E),$$

with equality if and only if Ω is a disk.

Actually, this ”new” conjecture is not really new...

Proposition

If for all $E > 0$, Faber-Krahn for H^{Ω}_{E} holds then Faber-Krahn holds for D^{Ω}.

Proof :

- If for all $E > 0$, Faber-Krahn for H^{Ω}_{E} holds. In $E = E_1(\Omega)$ there holds

$$\mu^{\mathbb{D}}\Big(\sqrt{\frac{|\Omega|}{\pi}}E_1(\Omega)\Big) \leq 0 \quad \text{and} \quad E_1(\mathbb{D}) \leq \sqrt{\frac{|\Omega|}{\pi}}E_1(\Omega).$$

Why the new conjecture ?

It is related to the Bossel-Daners inequality (Faber-Krahn for Robin Laplacian). Define for $E > 0$:

$$\lambda^{\Omega}_{\mathrm{Rob}}(E) := \inf_{u \in H^1(\Omega) \setminus \{0\}} \frac{\|\nabla u\|^2_{\Omega} + E \int_{\partial\Omega} |u|^2 ds}{\|u\|^2_{\Omega}}$$

Theorem

There holds

$$\frac{\pi}{|\Omega|} \lambda^{\mathbb{D}}_{\mathrm{Rob}}\left(\sqrt{\frac{|\Omega|}{\pi}} E\right) \leq \lambda^{\Omega}_{\mathrm{Rob}}(E),$$

with equality if and only if Ω is a disk.

M. H. Bossel:
C.R. Acad. Sci. Paris. (1986)

D. Daners:
Math. Ann. (2006)

The end !

Thank you for your attention !