A constructive approach for cross-point treatment in DDM for Helmholtz equation

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Numerical Waves - Nice

Universität Zürich[UZH]

The problem

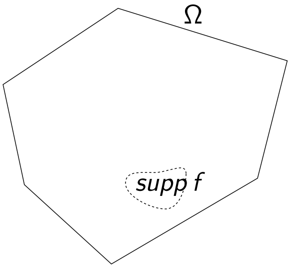
2D Helmholtz with Sommerfeld radiation condition at infinity

$$-\Delta u(\mathbf{x}) - \omega^2 u(\mathbf{x}) = f(\mathbf{x}) \quad \forall \mathbf{x} \in \mathbb{R}^2, \tag{1.1}$$

$$\lim_{\|\mathbf{x}\|\to\infty} \|\mathbf{x}\|^{1/2} \left(\nabla u(\mathbf{x}) \cdot \frac{\mathbf{x}}{\|\mathbf{x}\|} - \imath\omega u(\mathbf{x}) \right) = 0. \tag{1.2}$$

Computational domain Ω: **2nd order absorbing boundary conditions** on $\partial\Omega$

$$\partial_{\mathbf{n}} u - \imath\omega \left(1 + \frac{1}{2\omega^2}\partial_{\mathbf{t}\,\mathbf{t}}\right) u = 0 \dashrightarrow \left(1 - \frac{1}{2\omega^2}\partial_{\mathbf{t}\,\mathbf{t}}\right)\partial_{\mathbf{n}} u - \imath\omega u = 0$$



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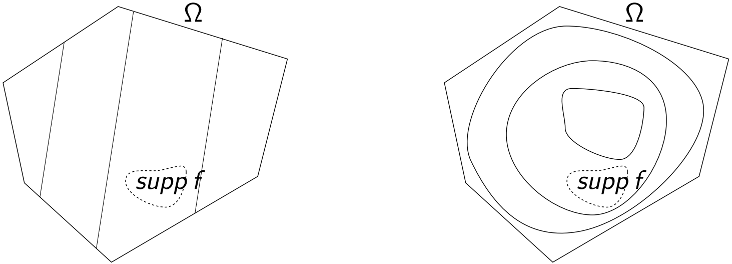
Domain decomposition $\Omega = \bigcup \Omega_i$:



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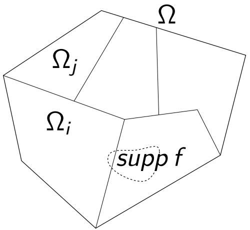
Polygonal domain decomposition $\Omega = \bigcup \Omega_i$:

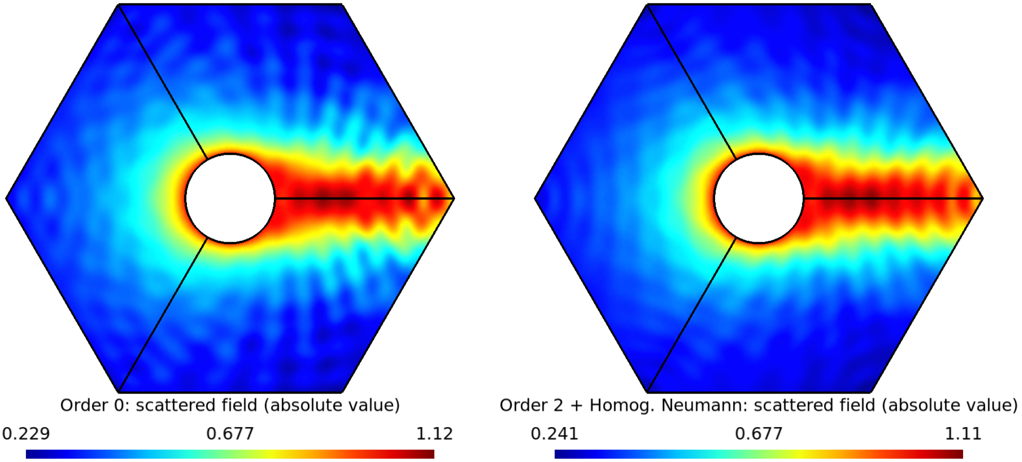


2nd order Robin type transmission conditions over $\partial\Omega_i \cap \partial\Omega_j$ that write

$$\partial_{\mathbf{n}_i} u_i - \imath\omega T u_i = -(\partial_{\mathbf{n}_j} u_j + \imath\omega T u_j) \tag{1.4}$$

with $T \simeq \left(1 - \frac{1}{2\omega^2}\partial_{\mathbf{t}\,\mathbf{t}}\right)^{-1}$

Artificial reflection without corner treatment



Quick overview

- Classical 2nd order ABC on a square in 2D: Joly, Lohrengel, Vacus (1999)
- DtN with right angle cross-points: Chaudet-Dumas, Gander
- Padé-type high-order ABC for right angles: Modave, Royer, Antoine, Geuzaine (2019, 2020)
- Multitrace nonlocal formalism: Claeys, Parolin (2019, 2020, 2021)

We chose to start from the coercive 2nd order ABC on straight lines

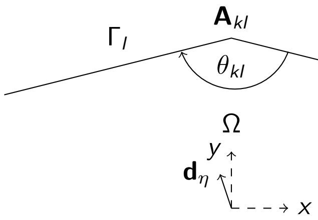
$$\left(1 - \frac{1}{2\omega^2}\partial_{\mathbf{t}\,\mathbf{t}}\right)\partial_{\mathbf{n}} u - \imath\omega u = 0 \tag{1.5}$$

and to seek for corner conditions complementing this equation that are also of order 2, hold for any angle and maintain the coercivity in the variational formulation of (1.5) with the objective of treating cross points using a similar frame.

A few comments :

- → we will work with transmission operators $T \neq T^*$
- → the algorithms will be endowed with decreasing energies ensuring convergence
- → reduction of the artificial reflection at corners with the new ABC
- → no detailed study of the convergence rate

Algebra



Normal/tangent vectors:

$$\begin{aligned} \mathbf{n}_k &= \left(\cos\frac{\theta_{kl}}{2}, -\sin\frac{\theta_{kl}}{2}\right) \\ \mathbf{t}_k &= \left(\sin\frac{\theta_{kl}}{2}, \cos\frac{\theta_{kl}}{2}\right) \end{aligned}$$

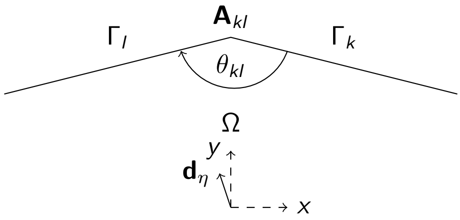
Outgoing tangent vectors $\boldsymbol{\tau}_k$ at $\mathbf{A}_{kl}$, incident plane wave

$$u_\eta(\mathbf{x}) = e^{\imath\omega \mathbf{d}_\eta \cdot \mathbf{x}}$$

Relations at corners $\mathbf{A}_{kl}$ to complement the relation

$$\left(1 - \frac{1}{2\omega^2}\partial_{\mathbf{t}_k \mathbf{t}_k}\right)\partial_{\mathbf{n}_k} u - \imath\omega u = 0, \quad \Gamma_k$$

Algebra



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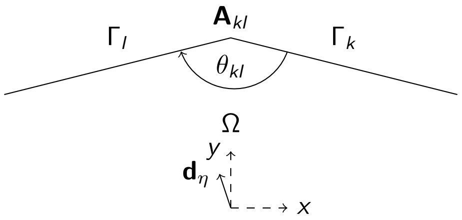
$$\left(1 - \frac{1}{2\omega^2}\partial_{\mathbf{t}_k\mathbf{t}_k}\right)\varphi_k - u_\eta = 0, \quad \Gamma_k$$

for $\varphi_k = (\imath\omega)^{-1}\partial_{\mathbf{n}_k} u$, verified by u_η up to $O(\eta^2)$

Algebra



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Relations at corners $\mathbf{A}_{kl}$ to complement the relation

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for $\varphi_k = (\imath\omega)^{-1}\partial_{\mathbf{n}_k}u$, verified by u_η up to $O(\eta^2)$
⇢ comes down to computing combinations of order 2 of the scalar products

$$\mathbf{n}_k\cdot\mathbf{d}_\eta \quad \text{and} \quad \boldsymbol{\tau}_k\cdot\mathbf{d}_\eta.$$

Second order relations

$$\partial_{\boldsymbol{\tau}_k}\varphi_k(\mathbf{A}_{kl}) - \frac{\imath\omega}{2}\left(\frac{\cos\theta_{kl}}{\cos\frac{\theta_{kl}}{2}} + \cos\frac{\theta_{kl}}{2}\right)\varphi_k(\mathbf{A}_{kl}) = \frac{\imath\omega}{2}\left(-\frac{\cos\theta_{kl}}{\cos\frac{\theta_{kl}}{2}} + \cos\frac{\theta_{kl}}{2}\right)\varphi_l(\mathbf{A}_{kl}) + O(\eta^2), \tag{2.1}$$

$$\partial_{\boldsymbol{\tau}_l}\varphi_l(\mathbf{A}_{kl}) - \frac{\imath\omega}{2}\left(\frac{\cos\theta_{lk}}{\cos\frac{\theta_{lk}}{2}} + \cos\frac{\theta_{lk}}{2}\right)\varphi_l(\mathbf{A}_{kl}) = \frac{\imath\omega}{2}\left(-\frac{\cos\theta_{lk}}{\cos\frac{\theta_{lk}}{2}} + \cos\frac{\theta_{lk}}{2}\right)\varphi_k(\mathbf{A}_{kl}) + O(\eta^2), \tag{2.2}$$

for all $\theta_{kl} \neq -\pi$ i.e. non-flat angles

Variational formulation of the ABC

Definition

Let $T : u \in L^2(\Gamma) \mapsto \varphi \in L^2(\Gamma)$ *where* $\varphi = (\varphi_k)_{k=0}^{K-1} \in \oplus_k H^1(\Gamma_k)$ *is the solution of*

$$\begin{aligned}&\sum_{k=0}^{K-1}\int_{\Gamma_k}\left(\varphi_k\overline{\psi_k} + \frac{1}{2\omega^2}\partial_{\mathbf{t}_k}\varphi_k\overline{\partial_{\mathbf{t}_k}\psi_k}\right)\mathrm{d}\gamma \\ &-\frac{1}{4\imath\omega}\sum_{\substack{k=0\\ l=k+1}}^{K-1}\left(\cos\frac{\theta_{kl}}{2}(\varphi_k+\varphi_l)\overline{(\psi_k+\psi_l)} + \frac{\cos\theta_{kl}}{\cos\frac{\theta_{kl}}{2}}(\varphi_k-\varphi_l)\overline{(\psi_k-\psi_l)}\right)(\mathbf{A}_{kl}) \\ &= \sum_{k=0}^{K-1}\int_{\Gamma_k} u\overline{\psi_k}\mathrm{d}\gamma \qquad \forall\psi\in\oplus_k H^1(\Gamma_k).\end{aligned} \tag{VF}$$

Series of properties: T is well-defined, $\|T\|_{\mathcal{L}(L^2(\Gamma))} \leq 1$, $T + T^*$ is positive self adjoint...
For Ω a K-sided regular polygonal domain approximating the disc of radius R as $K \to \infty$, this ABC converges towards

$$\left(1 - \frac{1}{2\omega^2 R^2}\partial_{\theta\theta} - \frac{\imath}{2\omega R}(1+\partial_{\theta\theta})\right)\partial_r u - \imath\omega u = 0.$$

Variational formulation of the global Helmholtz problem

Imposing $\partial_{\mathbf{n}} u = \imath\omega T(u)$ on $\partial\Omega$ in the weak fomulation of the Helmholtz equation gives:

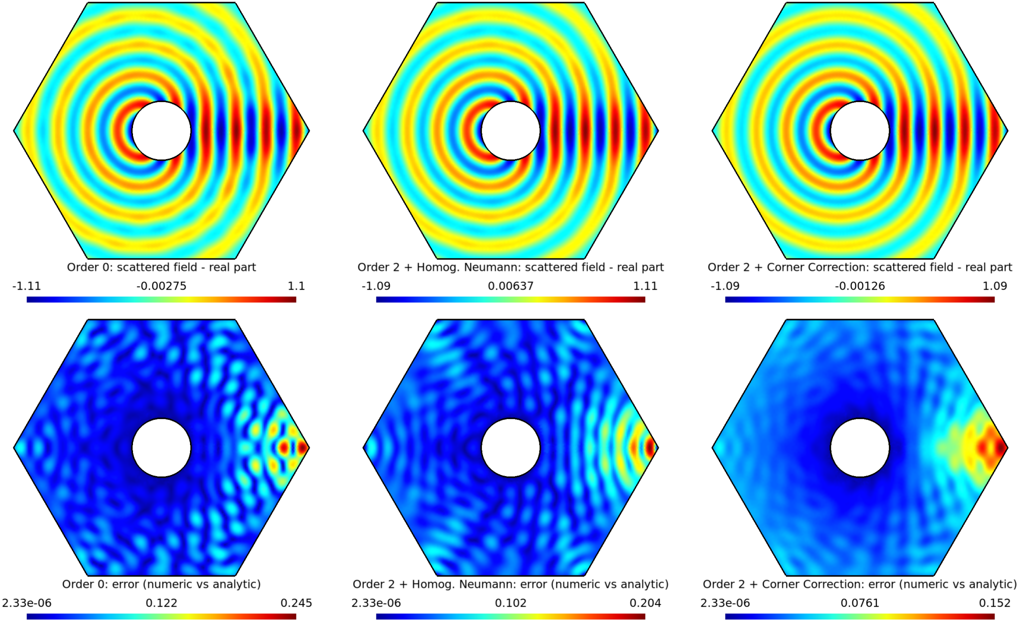
$$\boxed{\int_{\Omega} \left(\nabla u \cdot \overline{\nabla v} - \omega^2 u \overline{v}\right) \mathrm{d}\,\mathbf{x} - \imath\omega \int_{\partial\Omega} T(u)\overline{v}\mathrm{d}\gamma = \int_{\Omega} f\overline{v}\mathrm{d}\,\mathbf{x}, \qquad \forall v \in H^1(\Omega).} \tag{2.3}$$

(2.3) is well-posed: coercive+ compact, Fredholm's alternative apply and injectivity suffices to prove well-posedness. And if

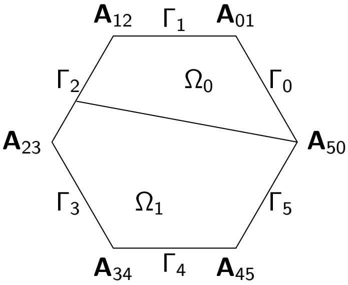
$$\int_{\Omega} \left(\nabla u \cdot \overline{\nabla v} - \omega^2 u \overline{v}\right) \mathrm{d}\,\mathbf{x} - \imath\omega \int_{\Gamma} T(u)\overline{v}\mathrm{d}\gamma = 0,$$

one has $\mathrm{Re} \int_{\Gamma} T(u)\overline{u} = 0 \Rightarrow T(u) = 0 \Rightarrow \partial_{\mathbf{n}} u = 0$, and the unique continuation principle implies $u = 0$.

ABC comparison



DDM



Algorithm (DDM-1) :

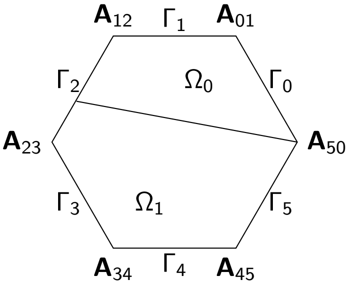
$$\begin{cases} -\Delta u_i^{p+1} - \omega^2 u_i^{p+1} = f, & \Omega_i, \\ \left(\partial_{\mathbf{n}^i} - \imath\omega\right) u_i^{p+1} = -\left(\partial_{\mathbf{n}^j} + \imath\omega\right) u_j^p, & \partial\Omega_i \cap \partial\Omega_j, \\ \partial_{\mathbf{n}} u_i^{p+1} = \imath\omega T(u^{p+1}), & \partial\Omega_i \cap \Gamma. \end{cases}$$

Lemma

The algorithm (DDM-1) with zero source term ($f = 0$) in endowed with a decreasing energy

$$E^p = \sum_i \int_{\partial\Omega_i \setminus \Gamma} |(\partial_{\mathbf{n}^i} - \imath\omega) u_i^p|^2$$

DDM



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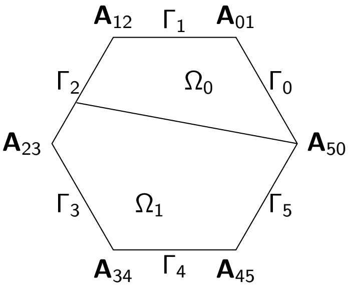
DDM



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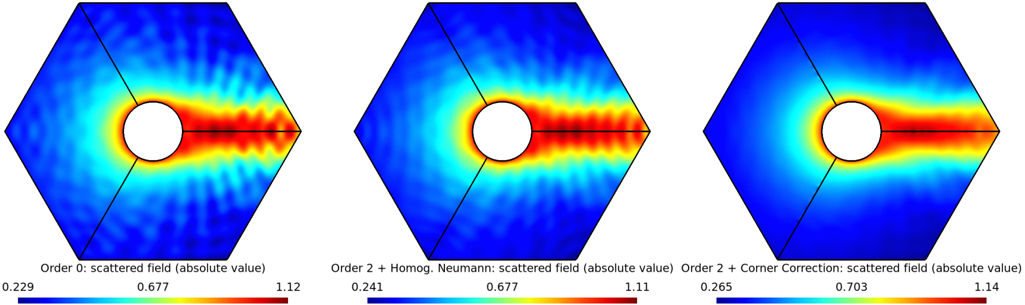
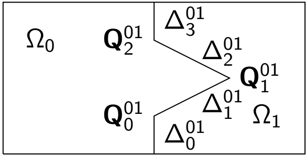
⇢ (DDM-2) decoupling subdomains and introducing auxiliary unknowns φ_k living on the Γ_k,
(DDM-3) decoupling the systems of equations on the u_i^{p+1} and on the φ_k^{p+1}.

Numerical illustration



Application to Robin transmission conditions of high order



Construct $T_{ij} \simeq (1 - \frac{1}{2\omega^2}\partial_{\mathbf{t}\,\mathbf{t}})^{-1}$ st

$$(\partial_{\mathbf{n}_i} - \imath\omega T_{ij})u_i = -(\partial_{\mathbf{n}_j} + \imath\omega T_{ij})u_j$$

on $\Delta^{ij} = \partial\Omega_i \cap \partial\Omega_j = \bigcup_{r=0}^{m_{ij}} \Delta_r^{ij}$.

Definition

Let $T_{ij} : u \in L^2(\Delta^{ij}) \mapsto \varphi \in L^2(\Delta^{ij})$ *where* $\varphi = (\varphi_r)_{r=0}^{m_{ij}} \in \oplus_r H^1(\Delta_r^{ij})$ *is the solution to*

$$\begin{aligned}
&\sum_{r=0}^{m_{ij}-1} \int_{\Delta_r^{ij}} \left(\varphi_r \overline{\psi_r} + \frac{1}{2\omega^2}\partial_{\mathbf{t}_i}\varphi_r \overline{\partial_{\mathbf{t}_i}\psi_r}\right) \mathrm{d}\gamma \\
&-\frac{1}{4\imath\omega} \sum_{\substack{r=0 \\ s=r+1}}^{m_{ij}-2} \left(\cos\frac{\theta_{rs}^{ij}}{2}(\varphi_r + \varphi_s)\overline{(\psi_r + \psi_s)} + \frac{\cos\theta_{rs}^{ij}}{\cos\frac{\theta_{rs}^{ij}}{2}}(\varphi_r - \varphi_s)\overline{(\psi_r - \psi_s)}\right)(\mathbf{Q}_r^{ij}) \\
&= \sum_{r=0}^{m_{ij}-1} \int_{\Delta_r^{ij}} u\overline{\psi_r}\mathrm{d}\gamma \qquad \forall \psi \in \oplus_r H^1(\Delta_r^{ij}).
\end{aligned} \tag{VF}$$

Properties and formulation

List of properties: $T_{ji} = T_{ij}^*$, $T_{ij} + T_{ji}$ is positive...
Define $T_i : L^2(\partial\Omega_i) \to L^2(\partial\Omega_i)$ on the closed boundary such that $T_i v|_{\Delta^{ij}} = T_{ij} v|_{\Delta^{ij}}$ and $T_i v|_{\partial\Omega_i \cap \partial\Omega} = v|_{\partial\Omega_i \cap \partial\Omega}$.

$$\boxed{\begin{array}{rcll} -\Delta u_i^{p+1} - \omega^2 u_i^{p+1} & = & f, & \Omega_i, \\ (\partial_{\mathbf{n}_i} - \imath\omega T_i)\, u_i^{p+1} & = & -\left(\partial_{\mathbf{n}_j} + \imath\omega T_j^*\right) u_j^p, & \Delta^{ij}, \\ (\partial_{\mathbf{n}_i} - \imath\omega T_i)\, u_i^{p+1} & = & 0, & \partial\Omega_i \cap \Gamma. \end{array}} \qquad \text{(DDM-TC)}$$

Lemma

The algorithm (DDM-TC) with zero source term ($f = 0$) is endowed to a decreasing energy

$$J^p = \sum_i \sum_j \int_{\Delta^{ij}} (T_{ij} + T_{ji})^{-1} \left(\partial_{\mathbf{n}_i} u_i^p - \imath\omega T_{ij} u_i^p\right) \overline{\left(\partial_{\mathbf{n}_i} u_i^p - \imath\omega T_{ij} u_i^p\right)}$$

Key property for the proof: the isometry $|||\partial_{\mathbf{n}_i} u_i + \imath\omega T_i^* u_i||| = |||\partial_{\mathbf{n}_i} u_i - \imath\omega T_i u_i|||$ in a given norm associated to the spectral decomposition of $T_i + T_i^*$, for u solution of (DDM-TC) with zero source term ($f = 0$).

Proof

$$\boxed{\begin{array}{rcll} -\Delta u_i^{p+1} - \omega^2 u_i^{p+1} & = & f, & \Omega_i, \\ (\partial_{\mathbf{n}^i} - \imath\omega T_i)\, u_i^{p+1} & = & -\left(\partial_{\mathbf{n}^j} + \imath\omega T_j^*\right) u_j^p, & \Delta^{ij}, \\ (\partial_{\mathbf{n}_i} - \imath\omega)\, T_i u_i^{p+1} & = & 0, & \partial\Omega_i \cap \Gamma. \end{array}} \quad \text{(DDM-TC)}$$

$$\begin{aligned} J^{p+1} &= \sum_i \sum_j \int_{\Delta^{ij}} (T_i + T_i^*)^{-1} \left(\partial_{\mathbf{n}_i} u_i^{p+1} - \imath\omega T_i u_i^{p+1}\right) \overline{\left(\partial_{\mathbf{n}_i} u_i^{p+1} - \imath\omega T_i u_i^{p+1}\right)} \\ &= \sum_i \sum_j \int_{\Delta^{ij}} (T_j + T_j^*)^{-1} \left(\partial_{\mathbf{n}_j} u_j^{p} + \imath\omega T_j^* u_j^{p}\right) \overline{\left(\partial_{\mathbf{n}_j} u_j^{p} + \imath\omega T_j^* u_j^{p}\right)} \\ &= \sum_j |||\partial_{\mathbf{n}_j} u_j^p + \imath\omega T_j^* u_j^p||| - \frac{1}{2} \int_{\partial\Omega_i \cap \partial\Omega} \left|(\partial_{\mathbf{n}_j} + \imath\omega) u_j^p\right|^2 \\ &= \sum_j |||\partial_{\mathbf{n}_j} u_j^p - \imath\omega T_j u_j^p||| - \frac{1}{2} \int_{\partial\Omega_i \cap \partial\Omega} \left|(\partial_{\mathbf{n}_j} + \imath\omega) u_j^p\right|^2 \\ &= \sum_j \sum_i \int_{\Delta^{ij}} (T_j + T_j^*)^{-1} \left(\partial_{\mathbf{n}_j} u_j^{p} - \imath\omega T_j u_j^{p}\right) \overline{\left(\partial_{\mathbf{n}_j} u_j^{p} - \imath\omega T_j u_j^{p}\right)} \\ &\quad + \frac{1}{2} \int_{\partial\Omega_i \cap \partial\Omega} \left|(\partial_{\mathbf{n}_j} - \imath\omega) u_j^p\right|^2 - \frac{1}{2} \int_{\partial\Omega_i \cap \partial\Omega} \left|(\partial_{\mathbf{n}_j} + \imath\omega) u_j^p\right|^2 \end{aligned}$$

Proof

$$\boxed{\begin{array}{rcll} -\Delta u_i^{p+1} - \omega^2 u_i^{p+1} &=& f, & \Omega_i, \\ (\partial_{\mathbf{n}^i} - \imath\omega T_i)\, u_i^{p+1} &=& -\left(\partial_{\mathbf{n}^j} + \imath\omega T_j^*\right) u_j^p, & \Delta^{ij}, \\ (\partial_{\mathbf{n}_i} - \imath\omega)\, T_i u_i^{p+1} &=& 0, & \partial\Omega_i \cap \Gamma. \end{array}} \qquad \text{(DDM-TC)}$$

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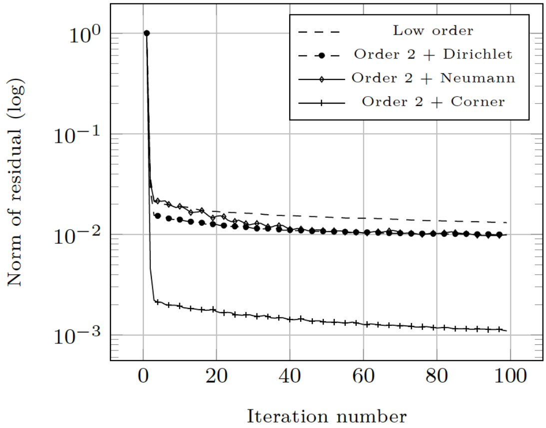
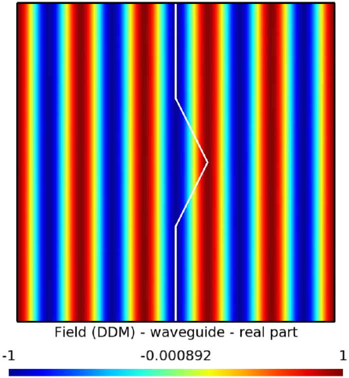
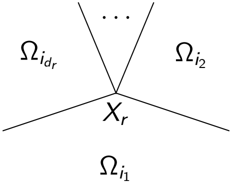
$$0 = \quad + \frac{1}{2} \int_{\partial\Omega_i \cap \partial\Omega} \left|(\partial_{\mathbf{n}_j} - \imath\omega) u_j^p\right|^2 - \frac{1}{2} \int_{\partial\Omega_i \cap \partial\Omega} \left|(\partial_{\mathbf{n}_j} + \imath\omega) u_j^p\right|^2 \quad < 0$$

Numerical illustration





Same structure for cross-points



Construct $T \simeq (1 - \frac{1}{2\omega^2}\partial_{\mathbf{t}\,\mathbf{t}})^{-1}$ st

$$(\partial_{\mathbf{n}} - \imath\omega T)u_\Sigma = -(\Pi\partial_{\mathbf{n}} + \imath\omega T\Pi)u_\Sigma$$

on $\Sigma = \cup_{ij}\Sigma_{ij} = \cup_{ij}\partial\Omega_i \cap \partial\Omega_j$, where Π is the natural exchange operator over the skeleton.

Definition

Let $T : u \in L^2(\Sigma) \mapsto \varphi \in L^2(\Sigma)$ where $\varphi \in H^1_{\mathrm{br}}(\Sigma)$ is the solution to

$$\sum_{i,j=1}^{N}\int_{\Sigma_{ij}}\left(\varphi_{ij}\overline{\psi_{ij}} + \frac{1}{2\omega^2}\partial_{\mathbf{t}_i}\varphi_{ij}\overline{\partial_{\mathbf{t}_i}\psi_{ij}}\right)\mathrm{d}\gamma + \frac{1}{2\omega^2}\sum_{r=1}^{N_X}(A^r\varphi_r,\psi_r)_{\mathbb{C}^{2d_r}} = (u,\psi)_{L^2(\Sigma)} \qquad \forall\psi \in H^1_{\mathrm{br}}(\Sigma),$$

for $A^r \in \imath\mathbb{R}^{2d_r\times 2d_r}$ a given skew-hermitian matrix.

Properties and formulation

T is well-defined, $\|T\|_{\mathcal{L}(L^2(\Sigma))} \leq 1$, $T + T^* > 0$, scalar product in $H^1_{\mathrm{br}}(\Sigma)$ associated to the spectral decomposition of $T + T^*$: $\langle u, v\rangle := \left(\left(\frac{T+T^*}{2}\right)^{-1} u, v\right)_{L^2(\Sigma)}$ which leads to $(H^1_T(\Sigma), |||\cdot|||) \sim (H^1_{\mathrm{br}}(\Sigma), \|\cdot\|_{H^1_{br}})$.

$$\boxed{\begin{array}{rcll} -\Delta u_i^{p+1} - \omega^2 u_i^{p+1} & = & f, & \Omega_i, \\ (\partial_{\mathbf{n}} - \imath\omega T)\, u_\Sigma^{p+1} & = & -\left(\Pi\partial_{\mathbf{n}} + \imath\omega T\Pi\right) u_\Sigma^p, & \Sigma, \\ (\partial_{\mathbf{n}} - \imath\omega)\, u_\Gamma^{p+1} & = & 0, & \Gamma. \end{array}} \qquad \text{(DDM-X)}$$

Lemma

Under the compatibility condition $T\Pi = \Pi T^*$, algorithm (DDM-X) with zero source term ($f = 0$) is endowed to a decreasing energy

$$F^p := |||(\partial_{\mathbf{n}} - \imath\omega T)u_\Sigma^p|||^2$$

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$$F^p := |||(\partial_{\mathbf{n}} - \imath\omega T)u_\Sigma^p|||^2 = |||(\partial_{\mathbf{n}} + \imath\omega T^*)u_\Sigma^{p-1}||| \leq F^{p-1} - 4\omega^2 \|u_\Gamma^{p-1}\|^2_{L^2(\Gamma)}$$

Key property:

$$|||(\partial_{\mathbf{n}} - \imath\omega T)u_\Sigma|||^2 + \|(\partial_{\mathbf{n}} - \imath\omega)u_\Gamma\|^2_{L^2(\Gamma)} = |||(\partial_{\mathbf{n}} + \imath\omega T^*)u_\Sigma|||^2 + \|(\partial_{\mathbf{n}} + \imath\omega)u_\Gamma\|^2_{L^2(\Gamma)}.$$

Compatibility condition: an admissible class of matrices A^r

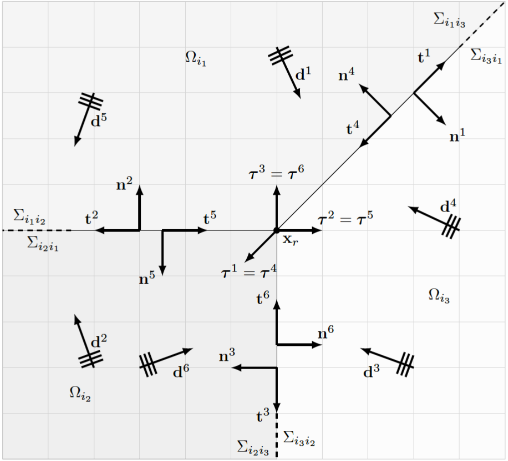
$u_{d^n}(\mathbf{x}) = \exp(\imath\omega d^n \cdot (\mathbf{x} - X_r))$ with d^n, d^{n+d_r} lin indep for all n plugged in the TC give

$$\operatorname{diag}(A^r((\mathbf{d}^m,\mathbf{n}^n))_{n,m=1}^{2d_r}) = \imath\omega \begin{pmatrix} (\mathbf{d}^n,\mathbf{n}^n)(\mathbf{d}^n,\mathbf{t}^n))_{n=1}^{d_r} \\ (-(\mathbf{d}^{n+d_r},\mathbf{n}^{n+d_r})(\mathbf{d}^{n+d_r},\mathbf{t}^{n+d_r}))_{n=1}^{d_r} \end{pmatrix} =: F_{r,\mathrm{pw}}$$

Considering the linear operators

$$\mathcal{L}_r : A \mapsto \begin{pmatrix} \operatorname{diag}(A((\mathbf{d}^m,\mathbf{n}^n))) \\ A - A^T \\ \Pi^r A^r + A^T \Pi^r \end{pmatrix}$$

can show that $(F_{r,\mathrm{pw}}, 0, 0)$ is orthogonal to $\ker \mathcal{L}_r^T$, so that there exists skew hermitian matrices $A^r \in \imath\mathbb{R}^{2d_r \times 2d_r}$ that verify the plane wave relation.



- DDM associated with a well posed variational formulations treating exterior (ABC) and interior (TC) corners and cross-points
- Decreasing energies endowed with each DDM, implying stability
- Mathematical analysis of this ABC and of the TC, but no convergence analysis

References:

- *Corners and stable optimized domain decomposition methods for the Helmholtz problem*, with B. Després and B. Thierry, under revision
- *Domain Decomposition Methods with optimized transmission conditions and cross-points*, with B. Després and B. Thierry, submitted

Numerical tests carried out using Gmsh and GetDP