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Localize resonances of open optical cavities

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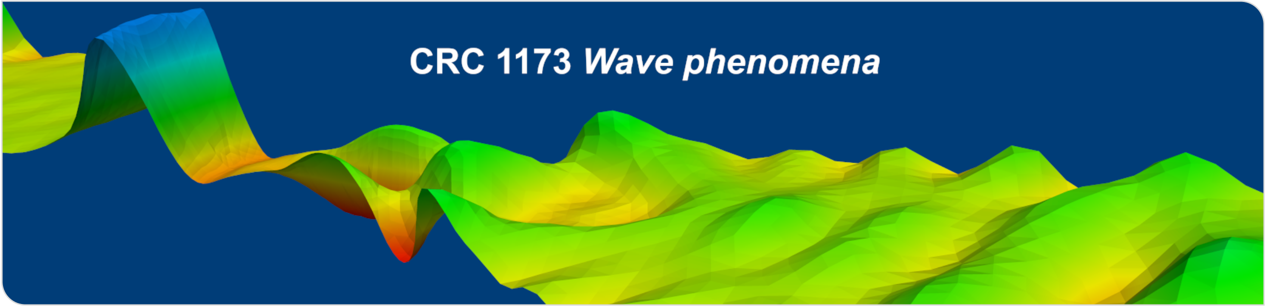
| 07 October 12021 HE



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3. Arbitrary classic cavity

4. Arbitrary metamaterial cavity

5. Conclusion

Transparent optical cavities

- Great interest in strongly confining and controlling light.
- Classic cavities made of dielectric material
  - $\varepsilon_c > 0$ (permittivity) and $\mu_c = 1$ (permeability)
  - Size: micro-scale
  - Applications: photonic, micro-resonator, sensing, etc.
- Metamaterial cavities made of metal
  - $\varepsilon_c < 0$ and $\mu_c = 1$
  - Size: nano-scale
  - Applications: sensing, antennas, high-resolution imaging, cloaking, etc.

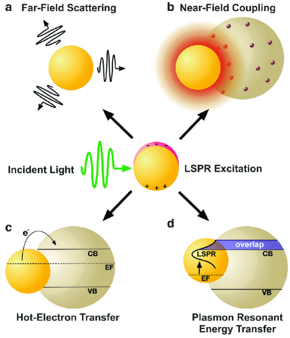
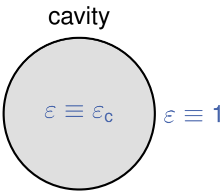


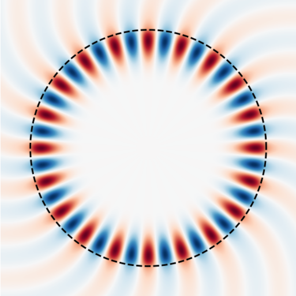
Figure: Erwin, Zarick, Talbert, and Bardhan, *Light trapping in mesoporous solar cells with plasmonic nanostructures*, Energy Environ. Sci., 2016.

Modelisation: (non lossy)
Cavity: 2D smooth bounded open set
PDE: Helmholtz equation

Localized resonances waves

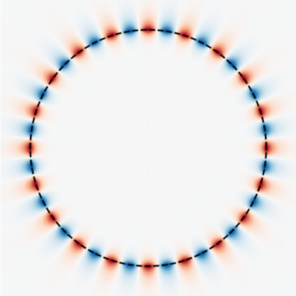


Classic
$\varepsilon_c = 2.25$



Whispering Gallery Mode

Metamaterial
$\varepsilon_c = -1.5$

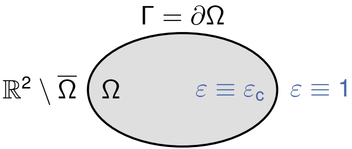


Surface Plasmon Wave

Scattering resonances of transparent cavities

$\varepsilon \neq 0$, piecewise smooth, discontinuous across Γ

- $\varepsilon \equiv \varepsilon_{\mathrm{c}}$ is $\mathscr{C}^\infty$ in the cavity $\overline{\Omega}$, $\varepsilon_{\mathrm{c}}(\gamma) \neq -1, 1$ for $\gamma \in \Gamma$
- $\varepsilon \equiv 1$ in $\mathbb{R}^2 \setminus \overline{\Omega}$



Resonances problem: Find $(\ell^2, u) \in \mathbb{C} \times \mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^2)$, $u \not\equiv 0$, such that

$$\begin{cases} -\operatorname{div}(\varepsilon^{-1}\,\nabla u) = \ell^2\, u & \text{in } \mathbb{R}^2 \\ [u]_\Gamma = 0 \quad \text{and} \quad \left[\varepsilon^{-1}\,\partial_n u\right]_\Gamma = 0 & \text{across } \Gamma \\ u \text{ is } \ell\text{-outgoing} & \end{cases}$$

u is ℓ-outgoing means that for $r > \sup_{x\in\Omega} |x|$ and $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, we have

$$u(r,\theta) = \sum_{m\in\mathbb{Z}} \Big(a_m\, \mathrm{H}^{(1)}_m(\ell r) + \underbrace{b_m}_{=0}\, \mathrm{H}^{(2)}_m(kr) \Big) \mathrm{e}^{\mathrm{i} m\theta}.$$

Scattering resonances of transparent cavities

$\varepsilon \neq 0$, piecewise smooth, discontinuous across Γ

- $\varepsilon \equiv \varepsilon_{\mathrm{c}}$ is $\mathscr{C}^{\infty}$ in the cavity $\overline{\Omega}$, $\varepsilon_{\mathrm{c}}(\gamma) \neq -1, 1$ for $\gamma \in \Gamma$
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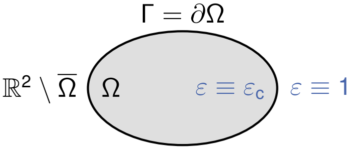

Resonances problem: Find $(\ell^2, u) \in \mathbb{C} \times \mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^2)$, $u \not\equiv 0$, such that

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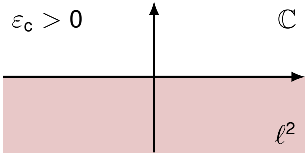

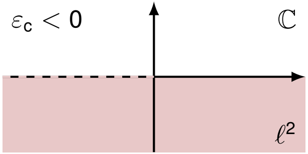

Known results

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Classic cavities $\varepsilon_c > 0$

- $-\operatorname{div}(\varepsilon^{-1}\nabla)$ is self-adjoint
- $\mathrm{sp}_{\mathrm{ess}} = \mathbb{R}_+$ and $\mathrm{sp}_{\mathrm{dis}} = \varnothing$
- Scattering by convex transparent obstacle, $\varepsilon_c < 1$ (non-trapping)
  - $\Im\ell$ tends to $-\infty$ when $\Re\ell \to +\infty$
  - $\exists$ a strip $\mathbb{R} \times [-a, 0]$ free of resonances below the real axis
- Scattering by convex transparent obstacle, $\varepsilon_c > 1$ (trapping)
  - Existence of sequences such that $\Im\ell \to 0$ super algebraically
  - Existence of Whispering Gallery Modes

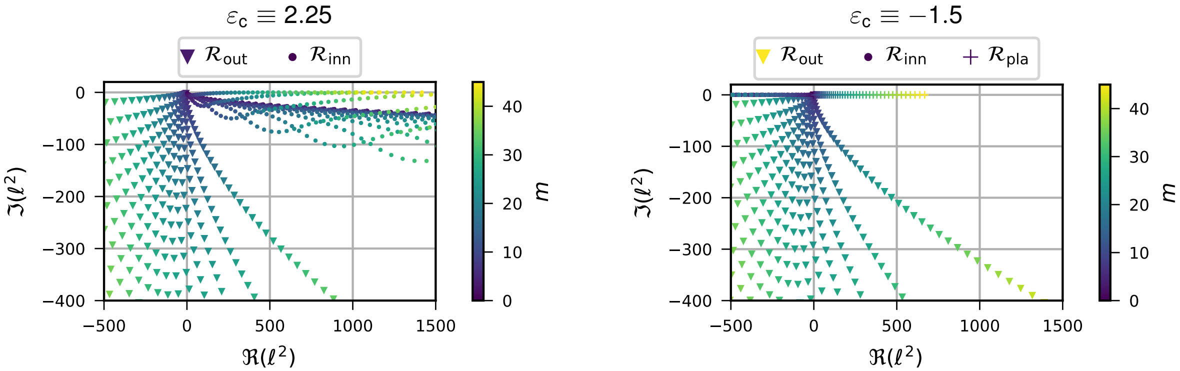
[Popov and Vodev 1999; Moiola and Spence 2019]

Metamaterial cavities $\varepsilon_c < 0$ and $\varepsilon_c|_\Gamma \neq -1$

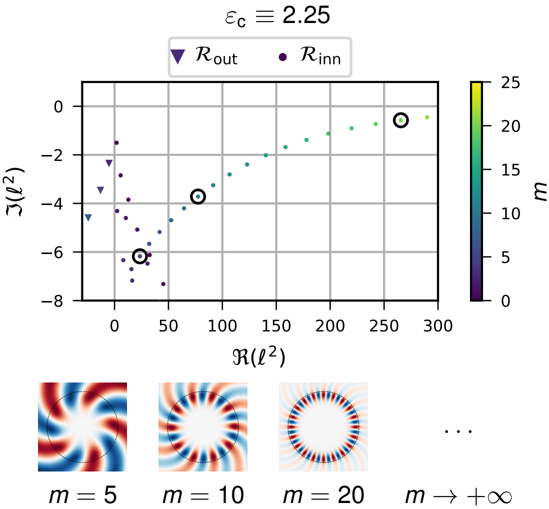
- $-\operatorname{div}(\varepsilon^{-1}\nabla)$ is self-adjoint
- $\mathrm{sp}_{\mathrm{ess}} = \mathbb{R}_+$ and $\varnothing \neq \mathrm{sp}_{\mathrm{dis}} \subset \mathbb{R}_-^*$

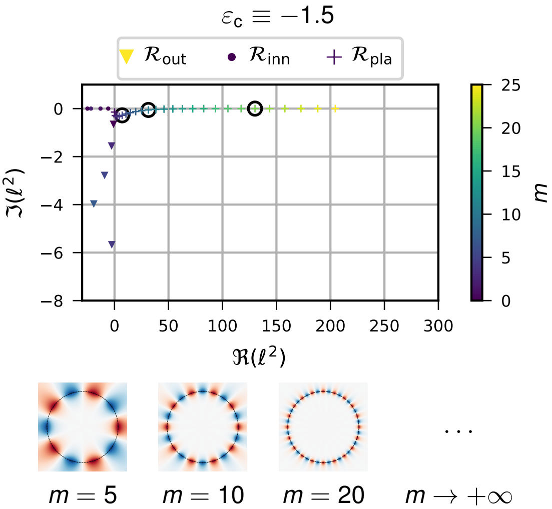
[Costabel and Stephan 1985; Carvalho and Moitier 2020]

Model problem: disk cavity with $\varepsilon_c \equiv$ cst

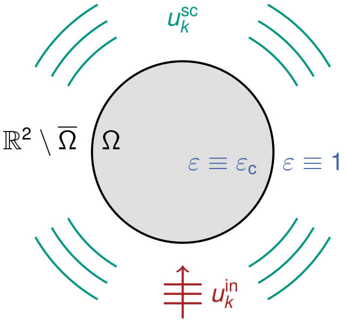


Model problem: disk cavity with $\varepsilon_c \equiv$ cst





Scattering by a 2D transparente cavity



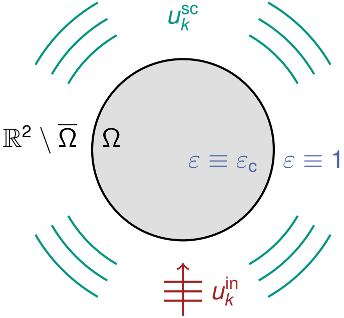
Given:

- a wavenumber $k > 0$
- an incident field $u_k^{in}(x, y) = e^{iky}$

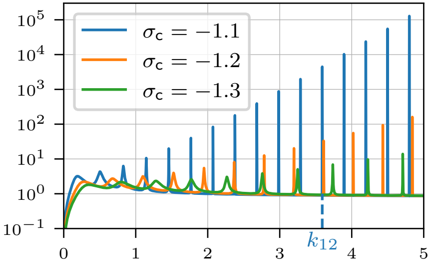
Find: the scattered field $u_k^{sc} \in H^1_{loc}(\mathbb{R}^2)$ such that

$$\begin{cases} -\operatorname{div}\left(\varepsilon^{-1}\nabla u^{sc}\right) - k^2 u^{sc} = k^2(1 - \varepsilon^{-1})u^{in} & \text{in } \mathbb{R}^2 \\ \left[\varepsilon^{-1}\partial_{\mathbf{n}} u^{sc}\right]_{\partial\Omega} = -\left[\varepsilon^{-1}\partial_{\mathbf{n}} u^{in}\right]_{\partial\Omega} & \text{across } \partial\Omega \\ u_k^{sc} \text{ is } k\text{-outgoing} & \end{cases}$$

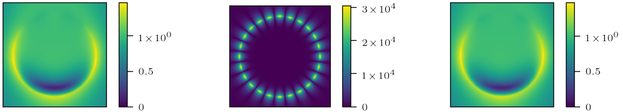
Scattering by a 2D transparente cavity



$$k \mapsto \frac{\|u^{\mathsf{sc}}\|_{L^2(B(0,2))}}{\|u^{\mathsf{in}}\|_{L^2(B(0,2))}}$$



$|u_k^{\mathsf{sc}}|$



k	$k_{12} - 0.01$	k_{12}	$k_{12} + 0.01$

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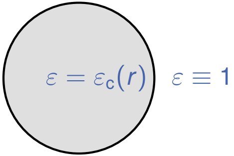
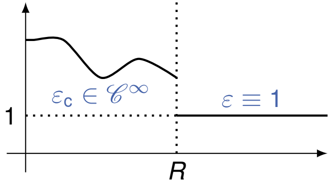
Angular Fourier Modes

Disk of radius R





Axisymmetry $\Longrightarrow$ angular Fourier decomposition: $u(r,\theta) = \sum_{m\in\mathbb{Z}} w_m(r)\, \mathrm{e}^{\mathrm{i}m\theta}$

$\Longrightarrow$ Family of problems indexed by $m \in \mathbb{Z}$: Find $(\ell, w) \in \mathbb{C} \times \mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^*_+, r\,\mathrm{d}r)$ such that $w \not\equiv 0$ and

$$\begin{cases} -\dfrac{1}{r}\left(\varepsilon(r)^{-1}\, r\, w'\right)' + \varepsilon(r)^{-1}\dfrac{m^2}{r^2}\, w = \ell^2\, w & \text{in } (0, +\infty) \\ [w]_{\{R\}} = 0 \quad \text{and} \quad \left[\varepsilon^{-1} w'\right]_{\{R\}} = 0 & \text{across } \{R\} \\ w(r) = \mathrm{H}^{(1)}_m(\ell r) & r > R \end{cases}$$

The Schrödinger Analogy

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We transform

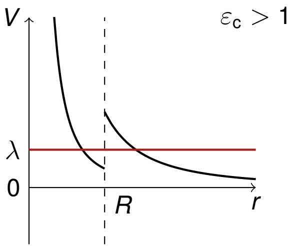
$$-\frac{1}{r}\left(\varepsilon(r)^{-1}\, r\, w'\right)' + \varepsilon(r)^{-1}\frac{m^2}{r^2}\, w = \ell^2\, w$$

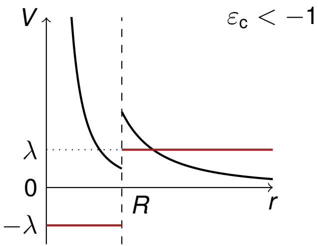
into

$$-h^2\, \mathscr{L} w + V\, w = \operatorname{sgn}(\varepsilon)\, \lambda\, w \qquad \text{where} \begin{cases} h = m^{-1} & \text{the semiclassical parameter} \\ \lambda = m^{-2}\ell^2 & \text{the spectral parameter} \\ \mathscr{L} w = \dfrac{1}{r}\left(|\varepsilon(r)|^{-1}\, r\, w'\right)' & \text{“Laplacian like”} \\ V(r) = \dfrac{1}{|\varepsilon(r)|\, r^2} & \text{the effective potential} \end{cases}$$

The Schrödinger Analogy

$$-h^2\,\mathscr{L}\,w + V\,w = \operatorname{sgn}(\varepsilon)\,\lambda\,w \qquad \text{where } V(r) = \frac{1}{|\varepsilon(r)|\,r^2}$$





The Schrödinger Analogy

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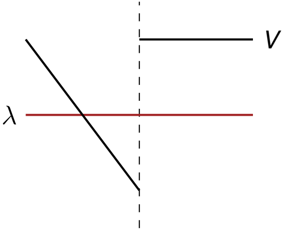
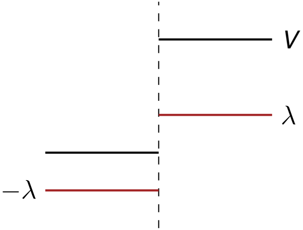


Local model



$$\frac{1}{R} + \frac{\varepsilon_c'(R)}{2\varepsilon_c(R)} > 0$$

Case $\varepsilon_{\mathrm{c}} < 0$: leading term as $h \to 0^+$

$\varepsilon_{\mathrm{c}}(R) = -\eta^2$

$$w^- = w^+$$

$$-\frac{1}{\eta^2}\partial_r w^- = \partial_r w^+$$

$$-h^2\mathscr{L}w^- + Vw^- = -\lambda w^-$$

$$-h^2\mathscr{L}w^+ + Vw^+ = \lambda w^+$$

Case $\varepsilon_c < 0$: leading term as $h \to 0^+$

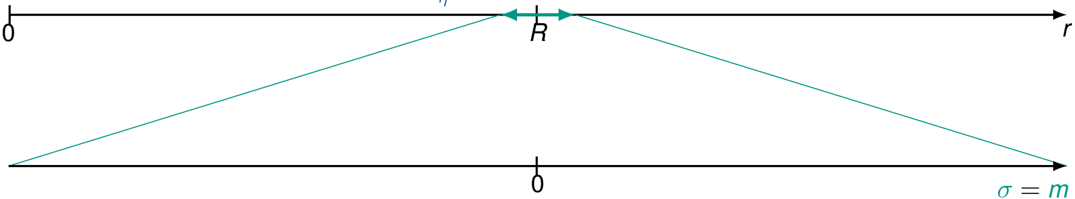
$\varepsilon_c(R) = -\eta^2$

$w^- = w^+$

$-h^2 \mathscr{L} w^- + V w^- = -\lambda w^-$

$-\frac{1}{\eta^2}\partial_r w^- = \partial_r w^+$

$-h^2 \mathscr{L} w^+ + V w^+ = \lambda w^+$



$\sigma = m\left(\frac{r}{R} - 1\right)$

$\varphi^{\pm}(\sigma) = w^{\pm}(r)$

Case $\varepsilon_c < 0$: leading term as $h \to 0^+$

$\varepsilon_c(R) = -\eta^2$

$$-h^2 \mathscr{L} w^- + V w^- = -\lambda w^-$$

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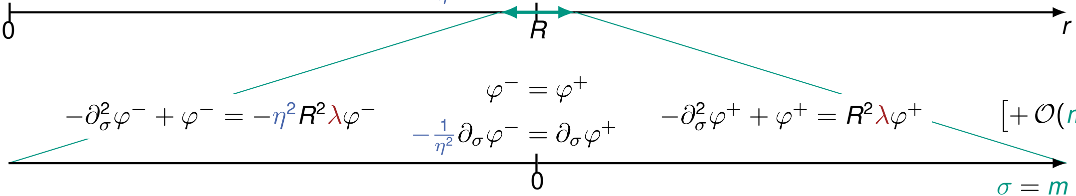

$$-\partial_\sigma^2 \varphi^- + \varphi^- = -\eta^2 R^2 \lambda \varphi^-$$

$$\varphi^- = \varphi^+$$

$$-\frac{1}{\eta^2}\partial_\sigma \varphi^- = \partial_\sigma \varphi^+$$

$$-\partial_\sigma^2 \varphi^+ + \varphi^+ = R^2 \lambda \varphi^+ \quad [+\mathcal{O}(m^{-1})]$$

$$\sigma = m\left(\frac{r}{R} - 1\right)$$

$$\varphi^\pm(\sigma) = w^\pm(r)$$

Case $\varepsilon_c < 0$: leading term as $h \to 0^+$

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0 — R — r

$$-\partial_\sigma^2 \varphi^- + \varphi^- = -\eta^2 R^2 \lambda \varphi^-$$

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$$-\partial_\sigma^2 \varphi^+ + \varphi^+ = R^2 \lambda \varphi^+$$

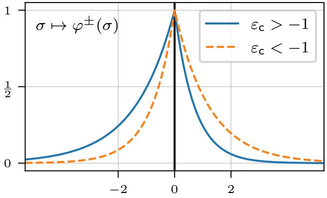
$$[+\mathcal{O}(m^{-1})]$$

$$\sigma = m\left(\frac{r}{R} - 1\right)$$

$$\varphi^\pm(\sigma) = w^\pm(r)$$

$$\lambda = R^{-2}\left(1 - \frac{1}{\eta^2}\right)$$

$$\varphi^\pm(\sigma) = \exp\left(-\eta^{\mp 1}\,|\sigma|\right)$$



Case $\varepsilon_c > 1$: leading term as $h \to 0^+$

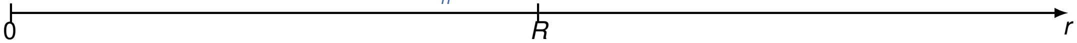
$$n^2 = \varepsilon_c(R), \quad \kappa = \frac{1}{R} + \frac{\varepsilon_c'(R)}{2\varepsilon_c(R)}$$

$$-h^2 \mathscr{L} w^- + V w^- = \lambda w^-$$

$$w^- = w^+$$

$$\frac{1}{n^2}\partial_r w^- = \partial_r w^+$$

$$-h^2 \mathscr{L} w^+ + V w^+ = \lambda w^+$$



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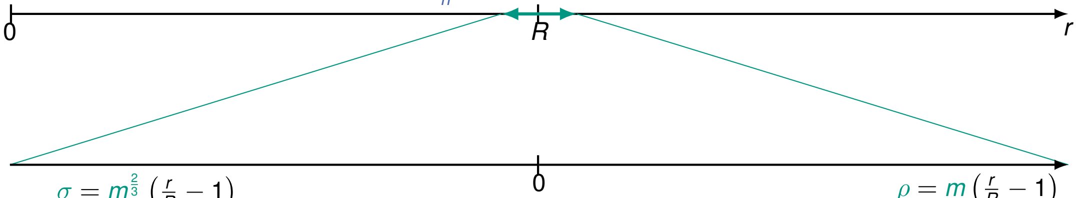

$$\sigma = m^{\frac{2}{3}}\left(\frac{r}{R} - 1\right)$$

$$\varphi(\sigma) = w^-(r)$$

$$\rho = m\left(\frac{r}{R} - 1\right)$$

$$\psi(\rho) = w^+(r)$$

Case $\varepsilon_c > 1$: leading term as $h \to 0^+$

$$n^2 = \varepsilon_c(R), \quad \kappa = \frac{1}{R} + \frac{\varepsilon_c'(R)}{2\varepsilon_c(R)}$$

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$$w^- = w^+$$

$$\frac{1}{n^2}\partial_r w^- = \partial_r w^+$$

$$-h^2 \mathscr{L} w^+ + V w^+ = \lambda w^+$$

0, R, r

$$-\partial_\sigma^2 \varphi - 2\kappa\,\sigma\,\varphi = \check{\lambda}\varphi$$

$$\varphi = \psi$$

$$0 = \partial_\rho \psi$$

$$-\partial_\rho^2 \psi - \frac{n^2-1}{n^2}\psi = 0 \qquad [+\mathcal{O}(m^{-1})]$$

0

$$\sigma = m^{\frac{2}{3}}\left(\frac{r}{R} - 1\right)$$

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$$w^- = w^+$$

$$\frac{1}{n^2}\partial_r w^- = \partial_r w^+$$

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0 — R — r

$$-\partial_\sigma^2 \varphi - 2\kappa\,\sigma\,\varphi = \check{\lambda}\varphi$$

$$\varphi = \psi$$

$$0 = \partial_\rho \psi$$

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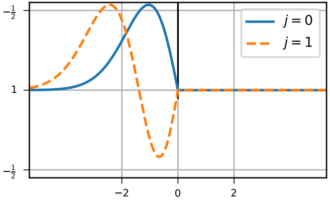
$$\sigma = m^{\frac{2}{3}}\left(\frac{r}{R} - 1\right)$$

$$\varphi(\sigma) = w^-(r)$$

$$\varphi(\sigma) = \mathrm{Ai}\left(-a_j - (2\kappa)^{\frac{1}{3}}\sigma\right)$$



$$\rho = m\left(\frac{r}{R} - 1\right)$$

$$\psi(\rho) = w^+(r)$$

$$\check{\lambda} = (2\kappa)^{\frac{2}{3}} a_j, \qquad \psi(\rho) = 0$$

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Asymptotic expansion of WGM resonances

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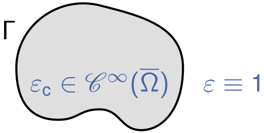


- m: number of curvilinear oscillations
- j: number of normal oscillations
- inner boundary layer $\sim m^{-\frac{2}{3}}$
- outer boundary layer $\sim m^{-1}$
- Taylor coefficient of W_j can be computed with a CAS

Hypothesis

$\varepsilon_c > 1$ and $\kappa + \left.\frac{\partial_n \varepsilon_c}{2\varepsilon_c}\right|_\Gamma > 0$ where κ is the curvature of Γ

Theorem

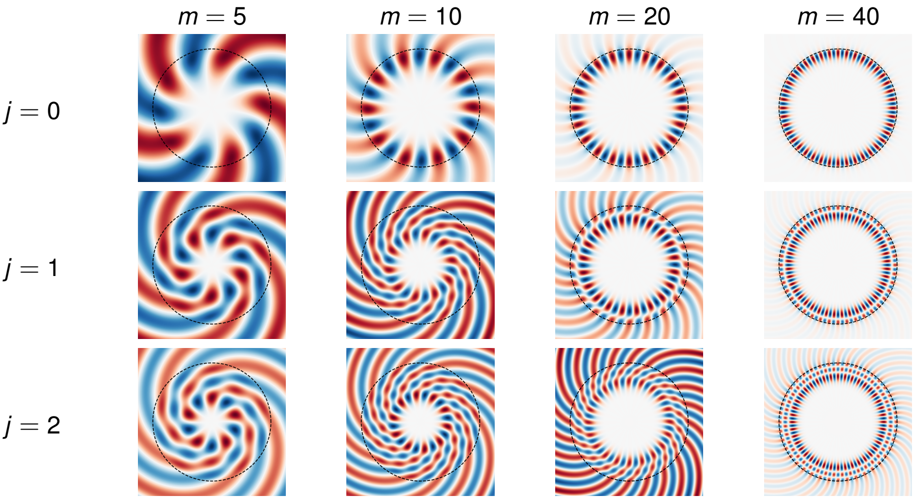
For $j \in \mathbb{N}$, as $m \to +\infty$, there exists resonances $\ell_{j,m}^2$ such that

$$\Re\left(\ell_{j,m}^2\right) = m^2\, W_j\left(m^{-1}\right) + \mathcal{O}\left(m^{-\infty}\right)$$
$$\Im\left(\ell_{j,m}^2\right) = \mathcal{O}\left(m^{-\infty}\right) \leq 0$$

where W_j are **real** functions with known Taylor expansion and

$$W_j(0) = \left(\frac{1}{2\pi}\int_\Gamma \sqrt{\varepsilon_c(\gamma)}\,\mathrm{d}\gamma\right)^{-2}.$$

Disk with $\varepsilon_c = 2.25$: Whispering Gallery Modes



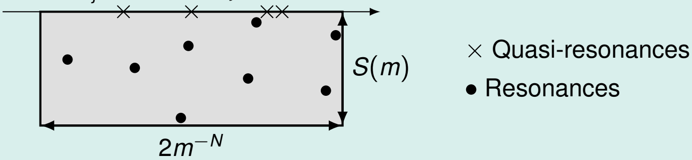
Asymptotic expansion of WGM resonances: Multiplicity

Theorem

There exists $S \in \mathscr{C}(\mathbb{R}_+)$ and $m_S > 0$ such that $S(m) = \mathcal{O}\left(m^{-\infty}\right)$ and for all $m > m_S$: for all $N \in \mathbb{N}$

$$\underline{\lambda}_j(m) = m^2 W_j(m^{-1})$$

in the box:
#Resonances $\geq$ 2 #Quasi-resonances.



We construct quasi-"eigenpairs" $\left(\underline{\lambda}_j(m), \underline{u}_j(m)\right)$ such that $-\operatorname{div}\left(\varepsilon^{-1}\nabla \underline{u}_j(m)\right) = \underline{\lambda}_j(m)\,\underline{u}_j(m) + \mathcal{O}\left(m^{-\infty}\right)$

$$\left(\underline{u}_i(m), \underline{u}_j(m)\right)_{\mathrm{L}^2(\mathbb{R}^2)} = \delta_{i,j} + \mathcal{O}\left(m^{-\infty}\right) \quad \text{and} \quad \left(\underline{u}_i(m), \overline{\underline{u}_j(m)}\right)_{\mathrm{L}^2(\mathbb{R}^2)} = \mathcal{O}\left(m^{-\infty}\right)$$

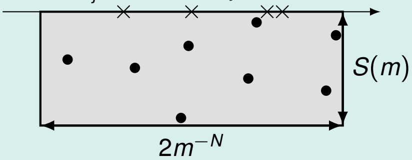
Asymptotic expansion of WGM resonances: Multiplicity

Theorem

There exists $S \in \mathscr{C}(\mathbb{R}_+)$ and $m_S > 0$ such that $S(m) = \mathcal{O}\left(m^{-\infty}\right)$ and for all $m > m_S$: for all $N \in \mathbb{N}$

$$\underline{\lambda}_j(m) = m^2 W_j(m^{-1})$$

in the box:
#Resonances $\geq$ 2 #Quasi-resonances.



$\times$ Quasi-resonances

$\bullet$ Resonances

Proof relie on Moitier 2019; Balac, Dauge, and Moitier 2021

- Construction of power series using WKB expansion and matched asymptotic
- Borel summation: This yields families of pairs that solve the true equation mod $\mathcal{O}\left(m^{-\infty}\right)$
- General results of *black box scattering* Tang and Zworski 1998; Stefanov 1999

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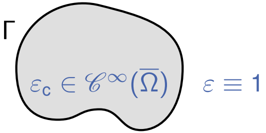
Asymptotic expansion of SPW resonances



- m: number of curvilinear oscillations
- inner and outer boundary layer $\sim m^{-1}$
- Taylor coefficient of P can be computed with a CAS
- if $\varepsilon_c < -1$ then $P(0) > 0$ and ℓ_m^2 are "true" resonances
- if $-1 < \varepsilon_c < 0$ then $P(0) < 0$ and ℓ_m^2 are negative eigenvalues

Hypothesis

$\varepsilon_c < -1$ or $-1 < \varepsilon_c < 0$

Theorem

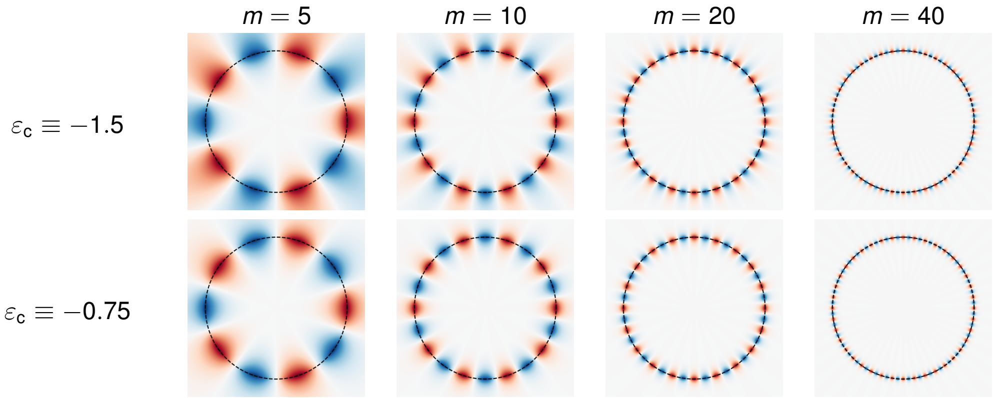
As $m \to +\infty$, there exists resonances ℓ_m^2 such that

$$\Re\left(\ell_m^2\right) = m^2 P\left(m^{-1}\right) + \mathcal{O}\left(m^{-\infty}\right)$$
$$\Im\left(\ell_m^2\right) = \mathcal{O}\left(m^{-\infty}\right) \leq 0$$

where P is a **real** function with known Taylor expansion and

$$P(0) = \left(\frac{1}{2\pi}\int_\Gamma \left(1 + \varepsilon_c(\gamma)^{-1}\right)^{-\frac{1}{2}} \mathrm{d}\gamma\right)^{-2}.$$

Disk with $\varepsilon_c < 0$: Surface Plasmon Waves

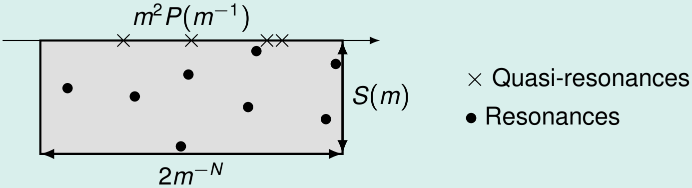


Asymptotic expansion of SPW resonances: Multiplicity

Theorem

There exists $S \in \mathscr{C}(\mathbb{R}_+)$ and $m_S > 0$ such that $S(m) = \mathcal{O}(m^{-\infty})$ and for all $m > m_S$: for all $N \in \mathbb{N}$

in the box:
#Resonances $\geq$ 2 #Quasi-resonances.



Proof relies on Carvalho and Moitier 2020; Mandel, Moitier, and Verfürth 2021

- Construction of power series using WKB expansion and matched asymptotic
- Borel summation: This yields families of pairs that solve the true equation mod $\mathcal{O}(m^{-\infty})$
- General results of *black box scattering* Tang and Zworski 1998; Stefanov 1999 (! not semi-bounded from below)

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Conclusion

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Conclusion

- We have characterize the WGM for the classic cavities and the SPW for metamaterial cavities.
- We have identify the width of the boundary layers.

Future works

- Imaginary part (! the potential is not analytic by opposition to Helffer and Sjöstrand 1986)
- Extend to Maxwell
- Cavities with corner

Conclusion

Conclusion

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Future works

- Imaginary part (! the potential is not analytic by opposition to Helffer and Sjöstrand 1986)
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Thank you for your attention

References I

[1] Stéphane Balac, Monique Dauge, and Zoïs Moitier. "Asymptotics for 2D whispering gallery modes in optical micro-disks with radially varying index". In: *IMA Journal of Applied Mathematics* (Aug. 2021). hxab033. ISSN: 0272-4960.

[2] Camille Carvalho and Zoïs Moitier. *Asymptotics for metamaterial cavities and their effect on scattering*. 2020. arXiv: 2010.07583.

[3] Martin Costabel and Ernst Stephan. "A direct boundary integral equation method for transmission problems". In: *J. Math. Anal. Appl.* 106.2 (1985), pp. 367–413. ISSN: 0022-247X.

[4] B. Helffer and J. Sjöstrand. "Résonances en limite semi-classique". In: *Mém. Soc. Math. France (N.S.)* 24-25 (1986), pp. iv+228. ISSN: 0037-9484.

[5] Rainer Mandel, Zoïs Moitier, and Barbara Verfürth. *Nonlinear Helmholtz equations with sign-changing diffusion coefficient*. 2021. arXiv: 2107.14516.

References II

[6] Andrea Moiola and Euan A. Spence. “Acoustic transmission problems: wavenumber-explicit bounds and resonance-free regions”. In: *Math. Models Methods Appl. Sci.* 29.2 (2019), pp. 317–354. ISSN: 0218-2025.

[7] Zoïs Moitier. “Étude mathématique et numérique des résonances dans une micro-cavité optique”. 2019REN1S053. PhD thesis. 2019.

[8] Georgi Popov and Georgi Vodev. “Resonances for transparent obstacles”. In: *Journées “Équations aux Dérivées Partielles” (Saint-Jean-de-Monts, 1999)*. Univ. Nantes, Nantes, 1999, Exp. No. X, 13.

[9] Plamen Stefanov. “Quasimodes and resonances: sharp lower bounds”. In: *Duke Math. J.* 99.1 (1999), pp. 75–92. ISSN: 0012-7094.

[10] Siu-Hung Tang and Maciej Zworski. “From quasimodes to resonances”. In: *Math. Res. Lett.* 5.3 (1998), pp. 261–272. ISSN: 1073-2780.