October 2021 – Numerical Waves, Nice

wavenumber-explicit hp-FEM analysis for the Helmholtz equation in heterogeneous media

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joint work with

M. Bernkopf (TU Wien), T. Chaumont-Frelet (Nice)

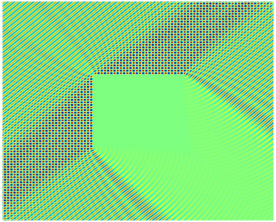
TU Wien
Institute of Analysis and Scientific Computing

Outline

Introduction

Heterogeneous Helmholtz equation

$$\begin{aligned} -\nabla \cdot (A(x)\nabla u) - k^2 n^2(x) u &= f \quad \text{in } \Omega, \\ b.c. & \quad \text{on } \partial\Omega, \\ \text{(radiation condition} & \quad \text{at } \infty) \end{aligned}$$



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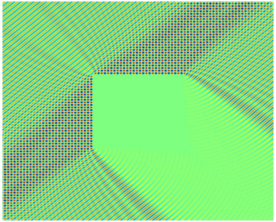

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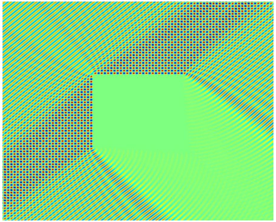

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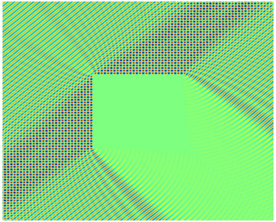

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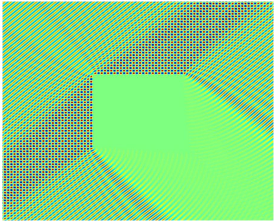

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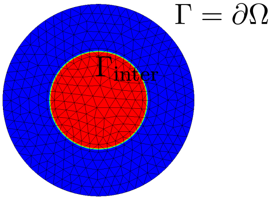
Ansatz $U(x,t) = e^{\mathbf{i}kt} u(x)$ yields $-k^2 u - \Delta u = f$

Heterogeneous Helmholtz problem

$$-\nabla \cdot (A(x)\nabla u) - k^2 n^2(x) u = f \qquad \text{in } \Omega,$$
$$\partial_{n_A} u - \mathbf{i} k u = g \qquad \text{on } \Gamma.$$

$$\partial_{n_A} u := \mathbf{n} \cdot (A \nabla u)$$



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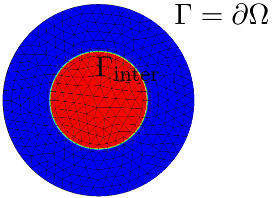
- $k \geq k_0 > 0$ is the wavenumber
- $f \in L^2(\Omega)$, $g \in H^{1/2}(\partial\Omega)$
- $A = A(x)$ piecewise analytic, uniformly SPD
- n piecewise analytic
- interface Γ_{inter} where A or n jumps is analytic



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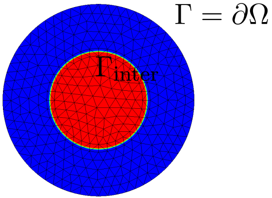

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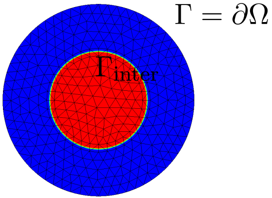

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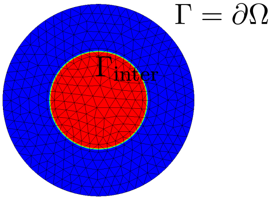

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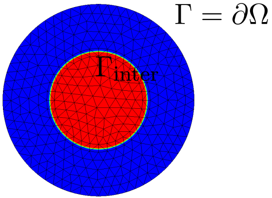

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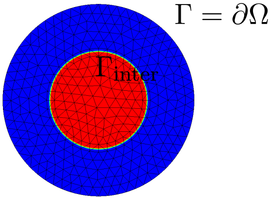

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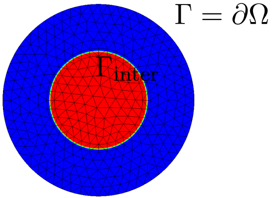

Generalization:

- boundary conditions $\partial_{n_A} u - T^- u = g$ (later)

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Weak formulation

$$\text{Find } u \in H^1(\Omega) \text{ s.t. } B(u,v) = F(v) \quad \forall v \in H^1(\Omega),$$

where

$$B(u,v) = \int_\Omega A(x)\nabla u \cdot \nabla \overline{v}\, dx - k^2 \int_\Omega n^2 u\overline{v}\, dx + \mathbf{i}k \int_{\partial\Omega} u\overline{v}\, ds,$$
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abstract FEM for $V_N \subset H^1(\Omega)$

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What is hp-FEM?

- approximation space V_N consists of piecewise polynomials of degree p

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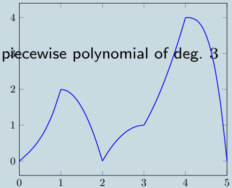
1D

- $\mathcal{T}$ = mesh on Ω
- $S^p(\mathcal{T}) := \{u \in C(\overline{\Omega}) \,|\, u|_K \in \mathcal{P}_p \quad \forall K \in \mathcal{T}\}$



- $N := \dim S^p(\mathcal{T}) \sim p|\mathcal{T}|$

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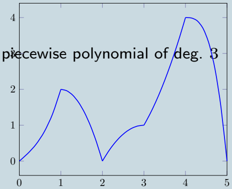

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  - h-FEM: refine the mesh $(h \to 0)$ for fixed p
  - p-FEM: $p \to \infty$ for fixed mesh
  - hp-FEM: simultaneously h-FEM and p-FEM

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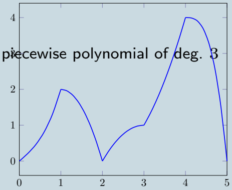

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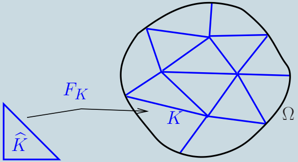
multi-D

- $\mathcal{T}$ = mesh on $\Omega \subset \mathbb{R}^d$ with element maps F_K
- $S^p(\mathcal{T}) := \{u \in H^1(\Omega) \,|\, u|_K \circ F_K \in \mathcal{P}_p \quad \forall K \in \mathcal{T}\}$
- $N := \dim S^p(\mathcal{T}) \sim p^d|\mathcal{T}|$
- h := mesh size = maximal element diameter



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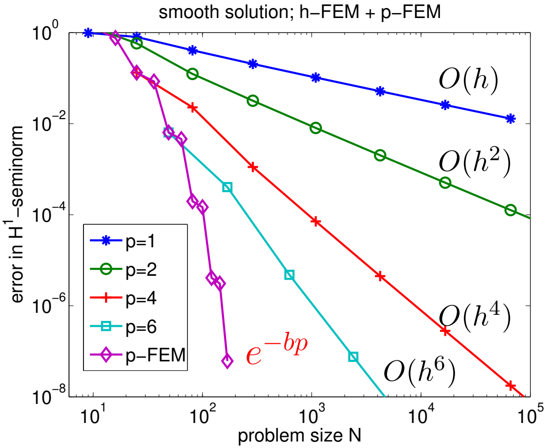
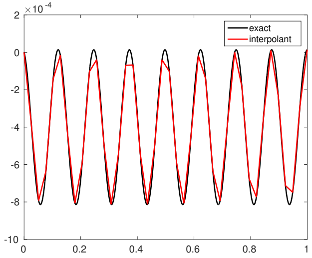
h-FEM vs. p-FEM for Poisson problem



- Poisson problem on $\Omega = (0,1)^2$
- smooth solution $u(x,y) = \sin(\pi x)\sin(\pi y)$
- FEM-convergence in energy norm:
  - h-FEM: $O(h^p)$
  - p-FEM: $O(e^{-bp})$

interpolation error and onset of quasioptimality ($p = 1$)



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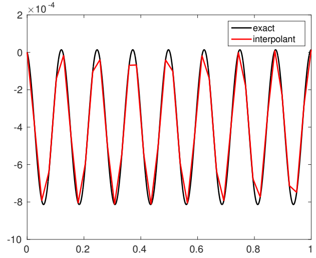

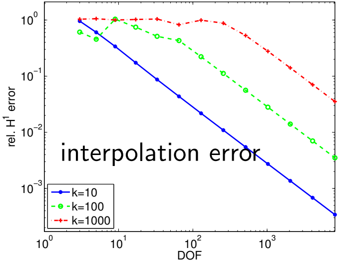

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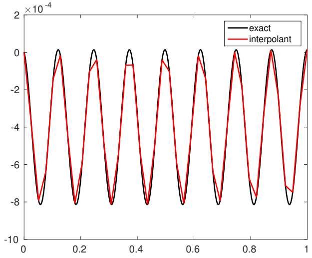

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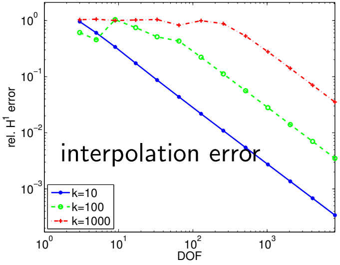

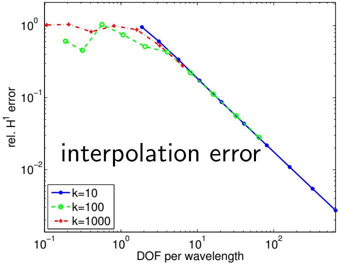

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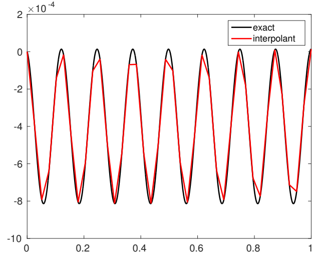

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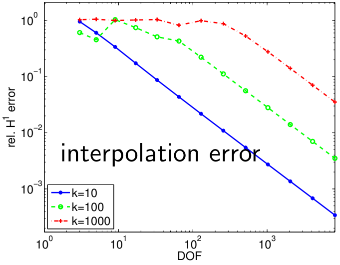
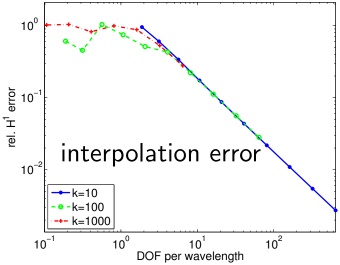
Observe: (interpolation) error per wavelength is independent of k





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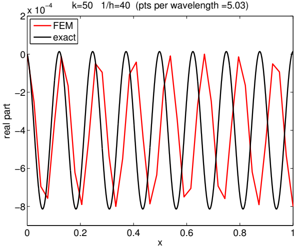

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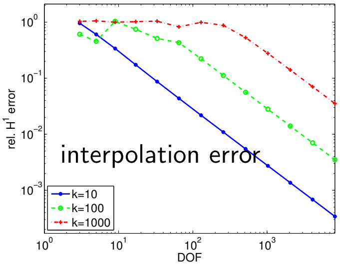

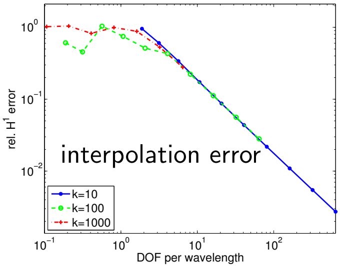

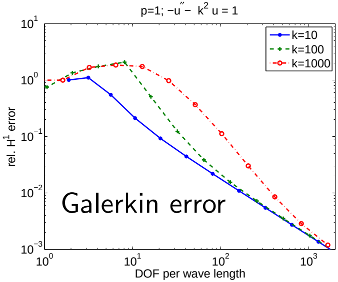

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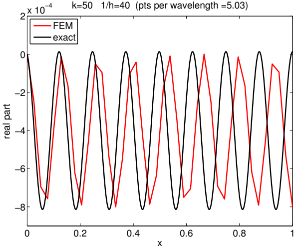
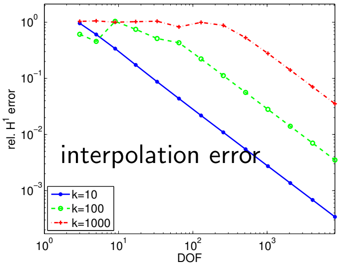
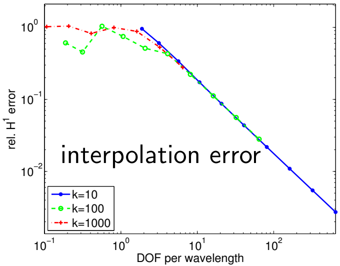
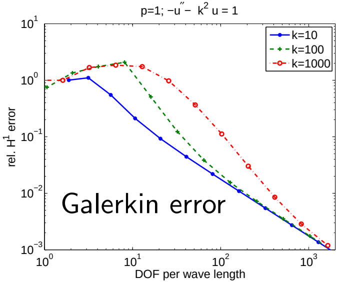


dispersion error:

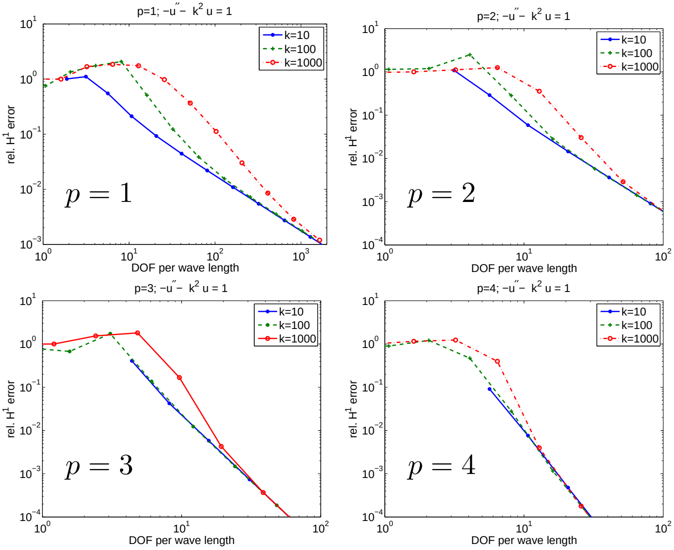
- fixed number of DOF per wavelength ("kh = constant") not sufficient to achieve given (rel.) accuracy
- Babuška & Sauter: this is the case for every 9-point stencil (in 2D)







Galerkin error, higher order methods



observation

- high order methods are less prone to dispersion errors

literature

- math. analysis on structured grids and homogeneous media: Babuška & Ihlenburg, Ainsworth, . . .,
- math. analysis on unstructured grids and homogeneous media: Melenk & Sauter
- h-version for heterogeneous media: Chaumont-Frelet & Nicaise

→ 2D results

Quasi-optimality of Galerkin hp-FEM

$$-\nabla\cdot(A\nabla u) - k^2 n^2 u = f \quad \text{in } \Omega, \qquad \partial_{n_A} u - \mathbf{i} k u = g \quad \text{on } \Gamma.$$

Theorem (Bernkopf, M. & Chaumont-Frelet, '21+)

Let Γ be analytic. Let A and n be piecewise analytic, with analytic interface Γ_{inter}. Assume polynomial well-posedness of the continuous problem. Let the mesh be aligned with Γ_{inter}. Given $c_2 > 0$ there exists c_1, independent of k, h, p such that under the scale resolution condition

$$\frac{kh}{p} \le c_1 \quad \text{and} \quad p \ge c_2(1+\log k)$$

the discretized problem is uniquely solvable, and there holds

$$\|u-u_N\|_{1,k} \lesssim \inf_{v_N \in S^p(\mathcal{T}_h)} \|u-v_N\|_{1,k},$$

where $\|v\|_{1,k}^2 := \|\nabla v\|_{L^2}^2 + k^2\|v\|_{L^2}^2$

Remark:

- onset of quasioptimality possible for $N = O(k^d)$, i.e., fixed number of DOF per wavelength, if $p = O(\log k)$ and $h = O(p/k)$
- alternative approach by Lafontaine, Spence, Wunsch '20 (for smooth coefficients)

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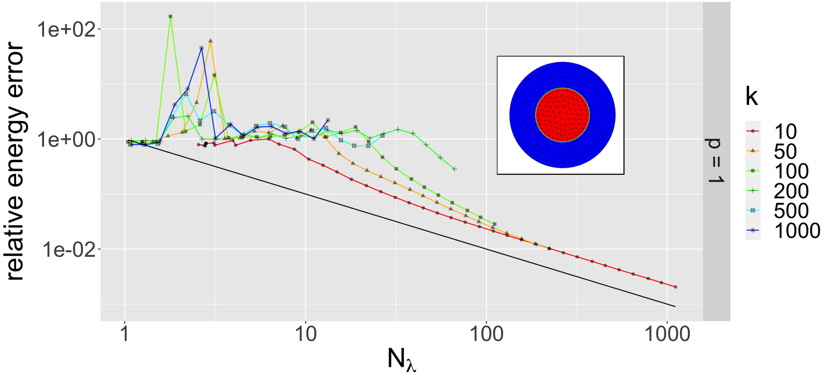
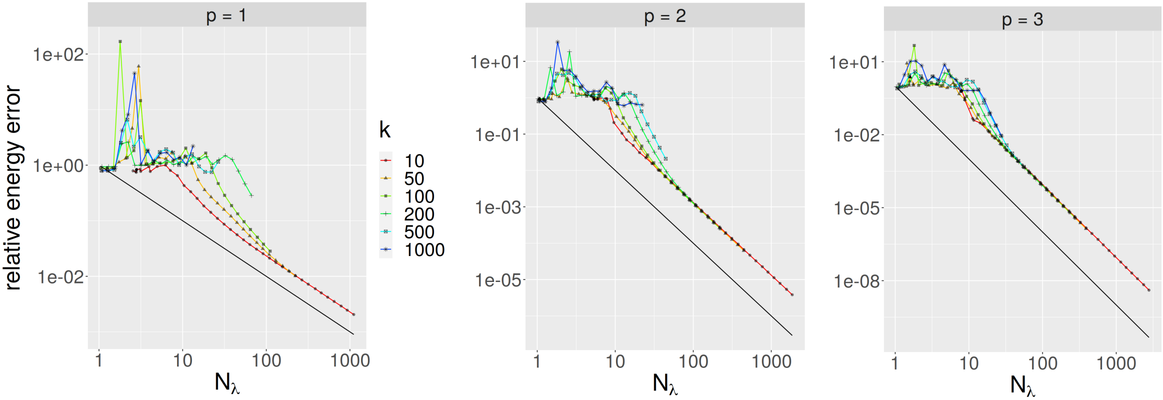
piecewise constant coefficients, analytic interface, $p = 1$



$$N_\lambda = \frac{2\pi \sqrt[d]{\mathrm{DOF}}}{k \sqrt[d]{|\Omega|}}$$

- $-\Delta u - k^2 n^2 u = 1$ on Ω
- $n = 1$ for $r < 1/2$, $n = 2$ for $r > 1/2$
- $\partial_n u - \mathbf{i} k u = 0$ on Γ
- computations with NGSolve (Schöberl et al.)

piecewise constant coefficient, analytic interface, $p = 1, 2, 3$



Analysis

facts and assumptions on the continuous problem

$$B(u,v) = \int_\Omega A(x)\nabla u \cdot \nabla \overline{v} - k^2 \int_\Omega n^2 u\overline{v} + \mathbf{i}k \int_{\partial\Omega} u\overline{v}, \qquad F(v) = \int_\Omega f\overline{v} + \int_{\partial\Omega} g\overline{v}$$

- norm: $\|u\|_{1,k}^2 := \|\nabla u\|_{L^2}^2 + k^2\|u\|_{L^2}^2$
- continuity: $|B(u,v)| \lesssim \|u\|_{1,k}\|v\|_{1,k}$
- Gårding inequality: $\|u\|_{1,k}^2 \lesssim |B(u,u)| + k^2\|u\|_{L^2}^2$
- stability assumption/polynomial well-posedness:
  There exists $\theta \in \mathbb{R}$ such that the solution $u \in H^1(\Omega)$ satisfies

$$\|u\|_{1,k} \lesssim k^\theta \left[\|f\|_{L^2} + \|g\|_{L^2}\right]$$

Remark: For $n \equiv 1$ and $A \equiv \mathrm{I}$, the stability assumption holds with $\theta = 0$ (Spence et al.)

The Schatz argument for quasi-optimality

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Schatz argument, cont'd

$$\|e_N\|_{1,k}^2 \lesssim \|u - v_N\|_{1,k}\|e_N\|_{1,k} + k^2\eta^2\|e_N\|_{1,k}^2$$

adjoint approximability

- $(v, e_N)_{L^2} = B(v, \xi) \quad \forall v \in H^1(\Omega)$
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Theorem (quasi-optimality)

If $k^2\eta^2$ is sufficiently small, then

$$\|u - u_N\|_{1,k} \lesssim \inf_{v_N \in V_N} \|u - v_N\|_{1,k}$$

upshot:

- $\to$ quantify η
- $\to$ understand the regularity of the dual solution ξ
- note: ξ solves again a Helmholtz problem

Heterogeneous Helmholtz model problem

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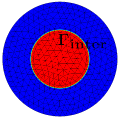

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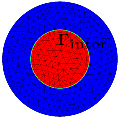

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regularity by decomposition: wavenumber explicit splitting

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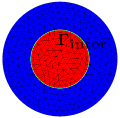

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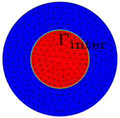

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regularity by decomposition: wavenumber explicit splitting

Heterogeneous Helmholtz model problem

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Theorem (Bernkopf, M. & Chaumont-Frelet, '21+)

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$\implies \eta \lesssim \dfrac{h}{p} + e^{-bp}k^{\theta} \implies$ scale resolution condition $\quad \dfrac{kh}{p}$ small and $p \gtrsim \log k$

key ingredients of the proof

- geometric series construction: $u = u_{0,H^2} + u_{0,\mathcal{A}} + u_{1,H^2} + u_{1,\mathcal{A}} + \cdots$
- use approximate solution operator/parametrix:
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frequency filters

- parameter $\sigma > 1$
- $\chi_{B_{\sigma k}}$ is the characteristic function of the ball $B_{\sigma k}(0)$
- Define the operators H_σ, $L_\sigma : L^2(\Omega) \to L^2(\mathbb{R}^d)$ by

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$$\|H_\sigma f\|_{0,\Omega} \le \|H_\sigma f\|_{0,\mathbb{R}^d} \overset{\text{Parseval}}{\sim} \|(1 - \chi_{B\sigma k})\mathcal{F}\mathcal{E} f\|_{0,\mathbb{R}^d} \le (\sigma k)^{-1} \| \, |\xi| \mathcal{F}\mathcal{E} f\|_{0,\mathbb{R}^d} \lesssim (\sigma k)^{-1} \|f\|_{1,\Omega}$$

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$=:u_{0,\mathcal{A}}$ p.w. analytic! (polyn. well-posedness)

$$-\nabla\cdot(A\nabla r) - k^2n^2r = 2k^2n^2u_{0,H^2} =: \tilde f \quad \text{in } \Omega,$$

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Contraction estimates (with $g = 0$):

$$\|\tilde f\|_0 \simeq k^2\|u_{0,H^2}\|_0 \lesssim k^2k^{-1}k^{-1+\varepsilon}\|H_\sigma f\|_{-\varepsilon} \lesssim k^\varepsilon(\sigma k)^{-\varepsilon}\|f\|_{0,\Omega} \lesssim \sigma^{-\varepsilon}\|f\|_{0,\Omega}$$

Goal: $S_k^-(f,g) = u = u_{0,H^2} + u_{0,\mathcal{A}} + S_k^-(\tilde f, \tilde g)$

$S_k^-(f,g) =: u$	$S_k^+(f,g) =: w$
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$$\Longrightarrow \|\tilde f\|_{0,\Omega} \le \frac{1}{2}\|f\|_{0,\Omega} \quad \text{for suitable } \sigma > 1$$

Underlying ingredients of the decomposition

$$L_k^- u = -\nabla \cdot (A\nabla u) - k^2 n^2 u \qquad L_k^+ u = -\nabla \cdot (A\nabla u) + k^2 n^2 u$$

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Boundary condition	T^-	T^+	$T^- - T^+$
Robin boundary	$\mathbf{i}ku$	0	$\mathbf{i}ku - 0 = k(\mathbf{i}u)$
Full Space, $\Gamma = \partial B_1(0)$	DtN_k	DtN_0	$\mathrm{DtN}_k - \mathrm{DtN}_0 = kR_0$
Full Space, Γ arbitrary	DtN_k	DtN_0	$\mathrm{DtN}_k - \mathrm{DtN}_0 = kR_0 + \mathcal{A}$
Second order ABCs	$\alpha\Delta_\Gamma + kR_0$	$\alpha\Delta_\Gamma$	kR_0

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Additionally:

- PML (fixed layer width)
- Helmholtz equations with first order terms
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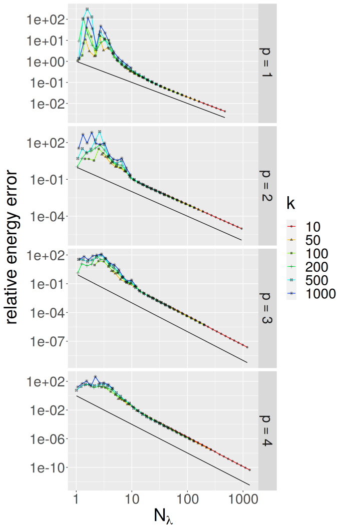
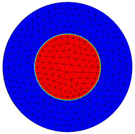
2^{nd} order ABC (Feng variant)



$$N_\lambda = \frac{2\pi \sqrt[d]{\mathrm{DOF}}}{k \sqrt[d]{|\Omega|}}$$



$$-\Delta u - k^2 n^2 u = f \quad \text{in } \Omega,$$

$$\partial_n - \alpha \Delta_\Gamma u - \beta u = g \quad \text{on } \Gamma,$$

$$\alpha = -\frac{\mathbf{i}}{2k}$$

$$\beta = \mathbf{i}k - \frac{1}{2} - \frac{\mathbf{i}}{8k}$$

- $n = 1$ for $r < 1/2$, $n = 2$ for $r > 1/2$
- $u(x, y) = \sin k(x + y)$
- computations with NGSolve (Schöberl et al.)

a FEM-BEM coupling strategy

FEM-BEM coupling with a mortar variable (3-field formulation)

- issue: the operator DtN_k is not explictly available
- available: the four "classical" BEM operators V_k, K_k, K'_k, W_k
- DtN_k can be represented as
- in principle, can represent $\mathrm{DtN}_k\, u$ by introducing new variable $\tilde{u}^m$ with $V_k \tilde{u}^m = \left(\frac{1}{2} - K_k\right) u$
- complication: V_k not invertible if k^2 is an eigenvalue of the interior Dirichlet problem
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- fact: $\tilde{u}^m = \partial_n u$. Instead, we will use $u^m := \partial_n u + \mathbf{i} k u$ as the auxiliary variable
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impedance trace as the mortar variable

coupling with impedance trace as the mortar variable, Mascotto, M., Perugia, Rieder '20

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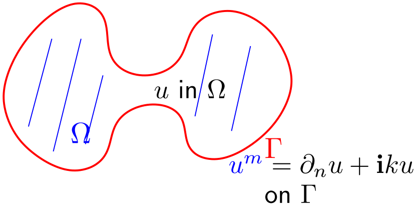

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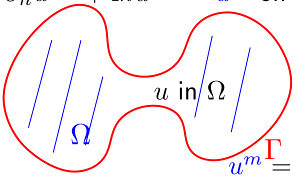
$(-\Delta - k^2)u^{\text{ext}} = 0$ in Ω^{ext}

$\partial_n u^{\text{ext}} + \mathbf{i}ku^{\text{ext}} = u^m$ on Γ



$u^m = \partial_n u + \mathbf{i}ku$ on Γ

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u - u^{\rm ext} &= 0 && \text{on } \Gamma.
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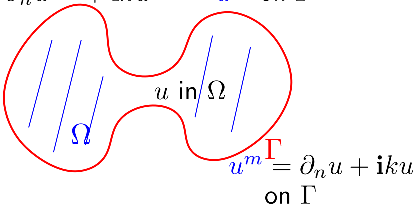

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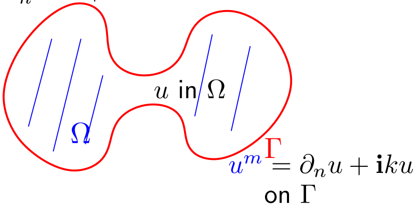

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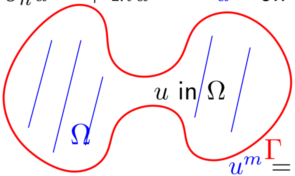

coupling with impedance trace as the mortar variable, Mascotto, M., Perugia, Rieder '20

$$\begin{aligned} -\nabla \cdot (A(x)\nabla u) - k^2 n^2(x) u = f & \quad \text{in } \Omega, \\ \partial_n u + \mathbf{i}ku - u^m = 0 & \quad \text{on } \Gamma, \\ u^{\text{ext}} - \mathcal{P}^+_{ItD} u^m = 0 & \quad \text{on } \Gamma, \\ u - \left[(\frac{1}{2} + K_k) u^{\text{ext}} - V_k (u^m - \mathbf{i}k u^{\text{ext}}) \right] = 0 & \quad \text{on } \Gamma. \end{aligned}$$

realization of $u^{\text{ext}} := \mathcal{P}^+_{ItD} u^m$ using combined field equations

$$\mathcal{B}_k u^{\text{ext}} + \mathbf{i}k \mathcal{A}'_k u^{\text{ext}} - \mathcal{A}'_k u^m = 0,$$

$$\mathcal{B}_k := -W_k - \mathbf{i}k \left(\frac{1}{2} - K_k \right), \qquad \mathcal{A}'_k := \frac{1}{2} + K'_k - \mathbf{i}k V_k$$

fact: $\mathcal{B}_k + \mathbf{i}k\mathcal{A}'_k$ invertible for all $k > 0$

The sesquilinear form

abbreviate: $\mathbf{u} := (u, u^m, u^{\text{ext}}) \in H^1(\Omega) \times H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$

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the case $k = 0$

$$\begin{aligned}\operatorname{Re} \mathcal{T}_0((u,u^m,u^{\text{ext}}),(u,u^m,u^{\text{ext}})) &= (A\nabla u, \nabla u)_{0,\Omega} + \langle W_0 u^{\text{ext}}, u^{\text{ext}}\rangle + \langle V_0 u^m, u^m\rangle \\ &\gtrsim |\nabla u|^2_{0,\Omega} + \|u^m\|^2_{-1/2,\Gamma} + |u^{\text{ext}}|^2_{1/2,\Gamma}\end{aligned}$$

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- $\rightsquigarrow$ Gårding inequality: exists compact operator Θ such that

$$\operatorname{Re}(\mathcal{T}_k(\mathbf{u},\mathbf{u}) + \langle \Theta\mathbf{u},\mathbf{u}\rangle) \geq \|\mathbf{u}\|^2_{E,k} := |u|^2_{1,\Omega} + k^2\|u\|^2_{0,\Omega} + \|u^m\|^2_{-1/2,\Gamma} + \|u^{\text{ext}}\|^2_{1/2,\Gamma}$$

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conforming discretization

- mesh $\mathcal{M}_h$ on Ω; aligned with the regions of smoothness of A, n, and with Γ
- trace mesh $\mathcal{M}_h^\Gamma := \mathcal{M}_h|_\Gamma$
- $S^{p,1}(\mathcal{M}_h) \subset H^1(\Omega)$ to discretize $u \in H^1(\Omega)$
- $S^{p-1,0}(\mathcal{M}_h^\Gamma) \subset L^2(\Gamma)$ to discretize $u^m \in H^{-1/2}(\Gamma)$
- $S^{p,1}(\mathcal{M}_h^\Gamma) \subset H^1(\Gamma)$ to discretize $u^{\text{ext}} \in H^{1/2}(\Gamma)$
- discrete space: $\mathbf{V}_h^{\text{conf}} := S^{p,1}(\mathcal{M}_h) \times S^{p-1,0}(\mathcal{M}_h^\Gamma) \times S^{p,1}(\mathcal{M}_h^\Gamma)$

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discrete formulation

Find $\mathbf{u}_h \in \mathbf{V}_h^{\text{conf}}$ s.t. $\mathcal{T}_k(\mathbf{u}_h, \mathbf{v}_h) = (f, v_h)_{0,\Omega} \qquad \forall \mathbf{v}_h \in \mathbf{V}_h^{\text{conf}}$

Theorem (Quasi-optimality of Galerkin hp-FEM: k-explicit analysis)

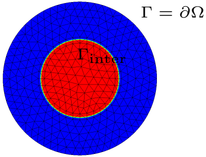
Let Γ be analytic, and let A and n be piecewise analytic, with analytic interface Γ_{inter}. Given $c_2 > 0$ there exists c_1, independent of k, h, p such that under the scale resolution condition

$$\frac{kh}{p} \leq c_1 \quad \text{and} \quad p \geq c_2(1 + \log k)$$

the discretized problem is uniquely solvable, and there holds

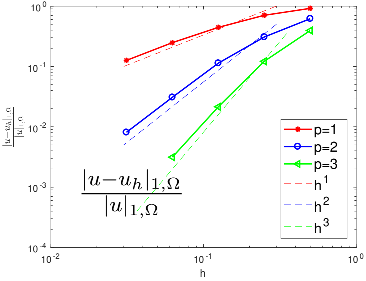
$$\|\mathbf{u} - \mathbf{u}_h\|_{E,k} \lesssim \inf_{\mathbf{v}_h \in \mathbf{V}_h^{\text{conf}}} \|\mathbf{u} - \mathbf{u}_h\|_{E,k}$$

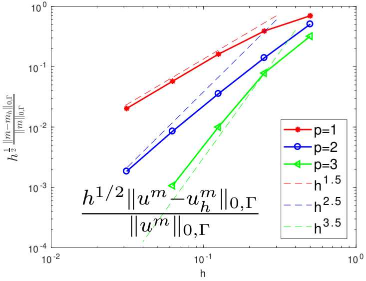
with implied constant independent of k.

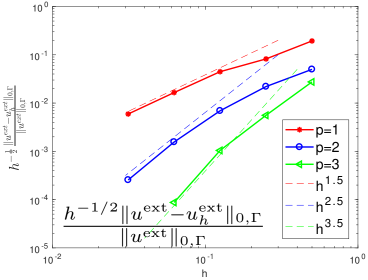


- $\mathbf{u} := (u, u^m, u^{\text{ext}})$
- $\|\mathbf{u}\|_{E,k}^2 := |u|_{H^1(\Omega)}^2 + k^2\|u\|_{L^2(\Omega)}^2 + \|u^m\|_{H^{-1/2}(\Gamma)}^2 + \|u^{\text{ext}}\|_{H^{1/2}(\Gamma)}^2$

mortar coupling, p.w. smooth coefficient, h-version





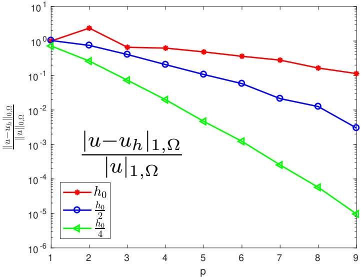


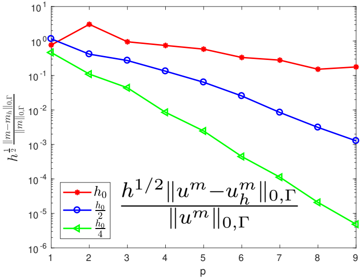
- $\Omega = (-0.5, 0.5)^3$
- $A(x) = 2$ on $(-0.2, 0.2)^3$, $\quad A(x) = 1$ on $\Omega \setminus (-0.2, 0.2)^3$
- $n \equiv 1$, $\qquad k = \sqrt{3}\pi$
- exact sol.: piecewise smooth

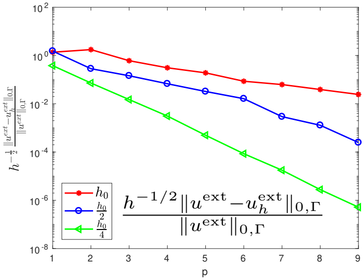
$$u(x,y,z) = \begin{cases} \sin^2\left(\frac{5\pi}{2}(x-0.2)\right)\sin^2\left(\frac{5\pi}{2}(y-0.2)\right)\sin^2\left(\frac{5\pi}{2}(z-0.2)\right) & \text{in } \Omega \\ \frac{e^{\mathbf{i}kr}}{r} & \text{in } \Omega^{\text{ext}} \end{cases}$$

- computations: NGSolve (Schöberl et al.), BEM++ (Betcke et al.), and H2Lib (Börm)

mortar coupling, p-version







- $\Omega = (-0.5, 0.5)^3$
- $A(x) = \mathrm{Id}$, $n \equiv 1$, $k = 3\sqrt{3}\pi$ (Dirichlet EV)
- exact sol.: piecewise analytic

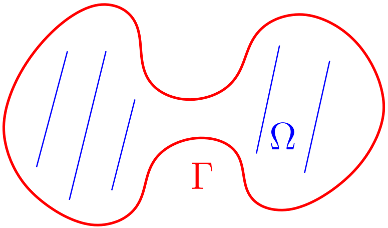
$$u(x,y,z) = \begin{cases} \sin(kx)\cos(ky) & \text{in } \Omega \\ \frac{e^{\mathbf{i}kr}}{r} & \text{in } \Omega^{\text{ext}} \end{cases}$$

- computations: NGSolve (Schöberl et al.), BEM++ (Betcke et al.), and H2Lib (Börm)

Maxwell

model problem: impedance problem in homogeneous media

$$\begin{aligned} \mathbf{curl}\,\mathbf{curl}\,\mathbf{u} - k^2\mathbf{u} &= \mathbf{f} && \text{in } \Omega, \\ (\mathbf{curl}\,\mathbf{u}) \times \mathbf{n} - \mathbf{i}k\mathbf{u}_T &= \mathbf{g}_T && \text{on } \Gamma, \end{aligned}$$



where $\Gamma := \partial\Omega$ is analytic

weak formulation:

Find $\mathbf{u}$ such that

$$a(\mathbf{u},\mathbf{v}) := (\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v})_{L^2(\Omega)} - ((\mathbf{u},\mathbf{v})) = \ell(\mathbf{v}) \quad \forall \mathbf{v}$$

$$((\mathbf{u},\mathbf{v})) := k^2(\mathbf{u},\mathbf{v})_{L^2(\Omega)} + \mathbf{i}k(\mathbf{u}_T, \mathbf{v}_T)_{L^2(\Gamma)}$$

- tangential component $\mathbf{v}_T = \mathbf{n} \times (\mathbf{v}|_\Gamma \times \mathbf{n})$
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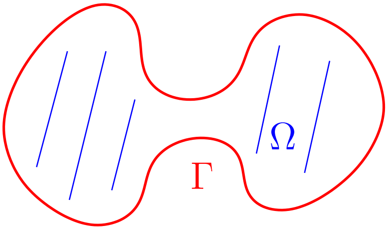

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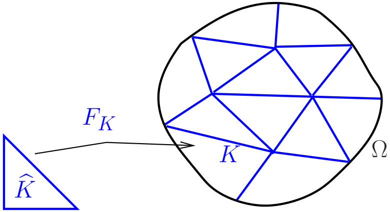
difference to Helmholtz

- Gårding inequality for $a(\cdot,\cdot)$ not (directly) available
- $\to$ need Helmholtz decompositions (continuous & and discrete)

$\mathbf{H}(\mathbf{curl})$-conforming discretization with hp-FEM

discretization

Find $\mathbf{u}_N \in \mathbf{X}_N$ such that $a(\mathbf{u}_N, \mathbf{v}) = \ell(\mathbf{v})$ for all $\mathbf{v} \in \mathbf{X}_N$



- $\mathcal{T}_h$ = triangulation (tetrahedra) of $\Omega \subset \mathbb{R}^3$, element maps F_K
- diam $K \sim h$ for all $K \in \mathcal{T}_h$
- $\mathbf{X}_N := \{\mathbf{u} \in \mathbf{H}(\mathbf{curl}, \Omega) \,|\, (F_K')^\top \mathbf{u} \circ F_K \in \mathcal{N}_p(\widehat{K})\}$
- $\mathcal{N}_p(\widehat{K}) := \{\mathbf{p}(\mathbf{x}) + \mathbf{x} \times \mathbf{q}(\mathbf{x}) \,|\, \mathbf{p}, \mathbf{q} \in (\mathcal{P}_p(\widehat{K}))^3\}$
- $N := \dim \mathbf{X}_N \sim h^{-3} p^3$

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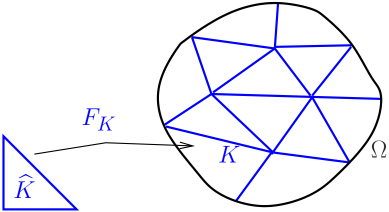

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H(curl)-conforming discretization with hp-FEM

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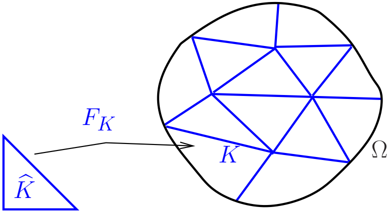

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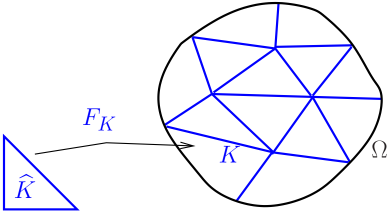

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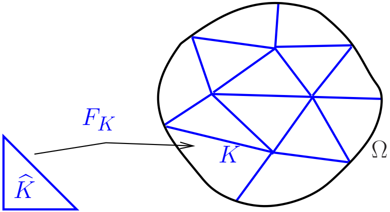

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quasi-optimality

Theorem (M.& Sauter '21+)

Let $\partial\Omega$ be analytic. Given $c_2 > 0$ there $\exists\ c_1, C > 0$ independent of k s.t. for

$$\frac{kh}{p} \leq c_1 \qquad \text{and} \qquad p \geq c_2 \log k$$

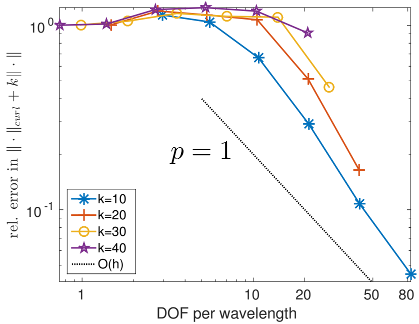
there holds

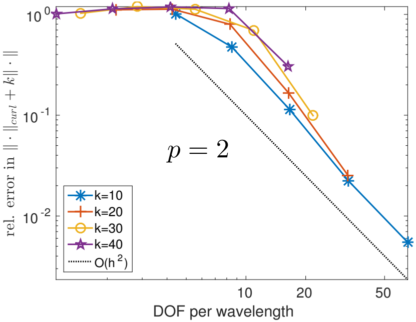
$$\|\mathbf{u} - \mathbf{u}_N\|_{\mathrm{imp},k} \leq C \inf_{\mathbf{v}_N \in \mathbf{X}_N} \|\mathbf{u} - \mathbf{v}_N\|_{\mathrm{imp},k},$$

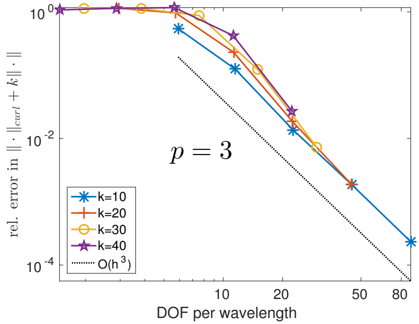
where $\|\mathbf{v}\|^2_{\mathrm{imp},k} := \|\,\mathbf{curl}\,\mathbf{v}\|^2_{L^2(\Omega)} + k^2\|\mathbf{v}\|^2_{L^2(\Omega)} + k\|\mathbf{v}_T\|^2_{L^2(\Gamma)}$

Remark

onset of quasi-optimality with problem size $N = O(k^3)$ (i.e., fixed number of DOF per wavelength) if $p = O(\log k)$ and $h = O(k/p)$



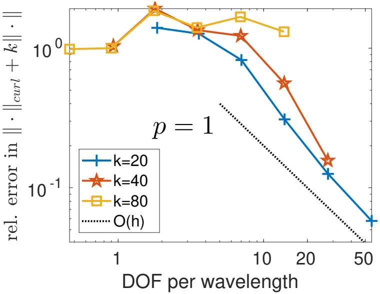


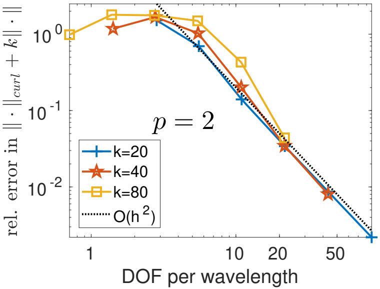


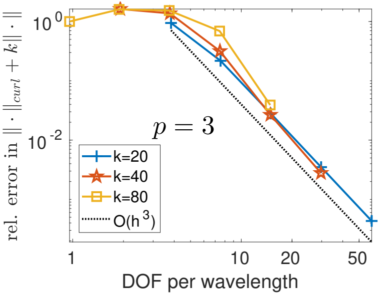
- smooth solution $\mathbf{u} = \mathbf{curl}(\sin(kx)(1,1,1)^T)$
- $\Omega = (-1,1)^3$
- Nédélec type II elements
- computations with: NgSolve (Schöberl *et al.*)
- largest problem size: $\sim 23m$

$\longrightarrow$ error vs. kh

perfectly conducting inclusion







- smooth solution

$$\mathbf{u} = \begin{pmatrix} 0 \\ -k\cos(kx)(x^2-0.25)(y^2-0.25)(z^2-0.25) \\ k\cos(kx)(x^2-0.25)(y^2-0.25)(z^2-0.25) \end{pmatrix}$$

- $\Omega = (-1,1)^3 \setminus [-1/2, 1/2]^3$
- Robin bc at outer bdy, Dirichlet at inner bdy
- Nédélec type II elements
- computations with: NgSolve (Schöberl *et al.*)
- largest problem size: $\sim 168m$ $(p = 2)$

$\rightarrow$ error vs. kh

conclusions and outlook

summary

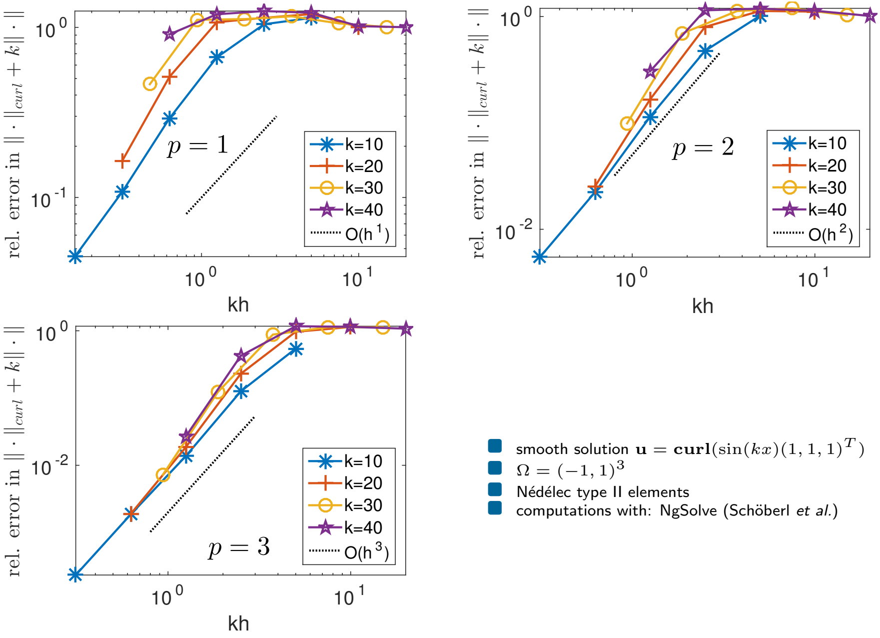
- quasi-optimality of hp-FEM for heterogeneous Helmholtz under resolution condition

$$\frac{kh}{p} \leq c_1 \quad \text{and} \quad p \geq c_2 \log k$$

- technique is fairly general and accommodates various boundary conditions:
  - Robin
  - exact DtN_k (arbitrary, analytic coupling boundary)
  - FEM-BEM coupling
  - 2^{nd} order ABCs
  - PML

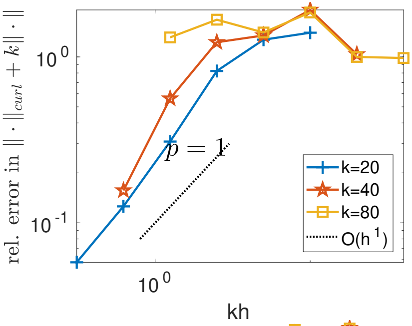
outlook

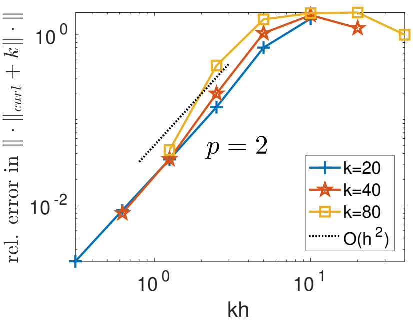
- corner domains (2D)
- heterogeneous time-harmonic elasticity equations (2D)
- heterogeneous time-harmonic Maxwell equations

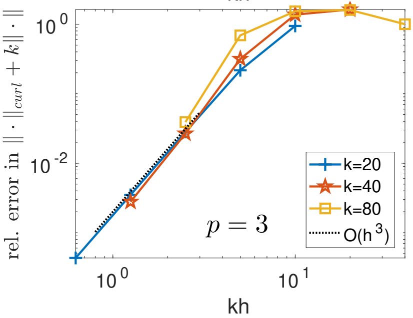


- smooth solution $\mathbf{u} = \mathbf{curl}(\sin(kx)(1,1,1)^T)$
- $\Omega = (-1,1)^3$
- Nédélec type II elements
- computations with: NgSolve (Schöberl *et al.*)

back

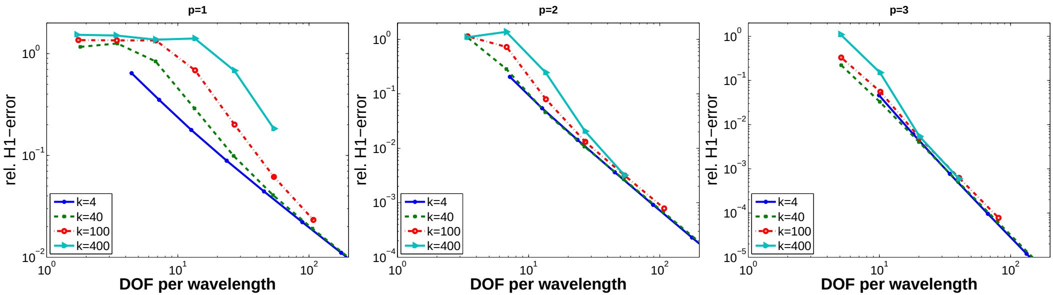






- smooth solution
$$\mathbf{u} = \begin{pmatrix} 0 \\ -k\cos(kx)(x^2-0.25)(y^2-0.25)(z^2-0.25) \\ k\cos(kx)(x^2-0.25)(y^2-0.25)(z^2-0.25) \end{pmatrix}$$
- $\Omega = (-1,1)^3 \setminus [-1/2,1/2]^3$
- Nédélec type II elements
- computations with: NgSolve (Schöberl *et al.*)

back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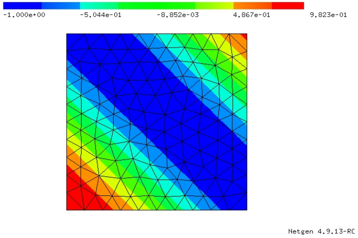
p=2
p=3
rel. H1−error
DOF per wavelength
k=4
k=40
k=100
k=400


-1.000e+00
-5.044e-01
-8.852e-03
4.867e-01
9.823e-01
Netgen 4.9.13-RC

back