High-frequency estimates on boundary integral operators for the Helmholtz exterior Neumann problem

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October 07, 2021 — Numerical Waves, Nice

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Considered problem

Scattering problem

- Solving Helmholtz equation $-\Delta u - k^2 u = 0$ in $\mathbb{R}^d \setminus \overline{\Omega}$, where Ω is an obstacle containing an open cavity, with particular attention to elliptic cavity.
- Plane wave $u^I(\mathbf{x}) = e^{ik\mathbf{d}\cdot\mathbf{x}}$ with $\mathbf{d} = [\cos(\theta), \sin(\theta), 0]$.

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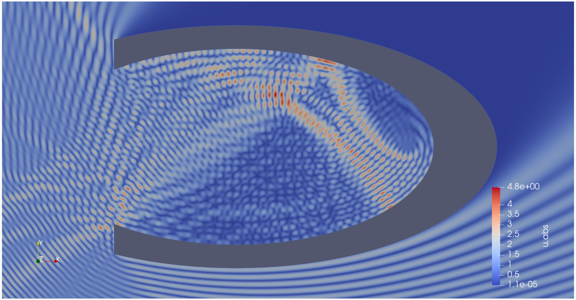

Figure 1: Absolute value of total field for $k = 122.473337808880$ and $\theta = \pi/4$

Fundamental solution

$$G_k(\mathsf{x}) := \frac{i}{4} H_0^{(1)}(k\|\mathsf{x}\|) \quad \text{for } \mathsf{x} \in \mathbb{R}^2 \setminus \{0\}, \quad \text{and} \quad \frac{e^{ik\|\mathsf{x}\|}}{4\pi\|\mathsf{x}\|} \quad \text{for } \mathsf{x} \in \mathbb{R}^3 \setminus \{0\},$$

Integral representation theorem

$$\int_{\partial\Omega} \mathsf{n}(\mathsf{y}) \cdot \nabla_{\mathsf{y}} G(\mathsf{x}-\mathsf{y}) u^s(\mathsf{y}) \, \mathrm{d}\sigma(\mathsf{y}) - \int_{\partial\Omega} G(\mathsf{x}-\mathsf{y}) \frac{\partial u^s}{\partial \mathsf{n}}(\mathsf{y}) \, \mathrm{d}\sigma(\mathsf{y}) = \begin{cases} u^s \text{ in } \mathbb{R}^d \setminus \overline{\Omega} \\ 0 \text{ in } \Omega \end{cases}$$

Boundary Integral Equations

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$$\mathcal{D}_k(u^S) - \mathcal{S}_k\left(\frac{\partial u^S}{\partial \mathbf{n}}\right) = \begin{cases} u^S \text{ in } \mathbb{R}^d \setminus \overline{\Omega} \\ 0 \text{ in } \Omega \end{cases}$$

$$\mathcal{D}_k(u^I) - \mathcal{S}_k\left(\frac{\partial u^I}{\partial \mathbf{n}}\right) = \begin{cases} 0 \text{ in } \mathbb{R}^d \setminus \overline{\Omega} \\ -u^I \text{ in } \Omega \end{cases}$$

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Total field

$$\mathcal{D}_k(u) - \mathcal{S}_k\left(\frac{\partial u}{\partial \mathbf{n}}\right) + u^I = u, \text{ in } \mathbb{R}^d \setminus \overline{\Omega}$$

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Dirichlet problem (sound-soft problem) $\gamma(u) = 0$

$$\gamma \circ \mathcal{S}_k\left(\frac{\partial u}{\partial \mathbf{n}}\right) = \gamma(u^I), \quad \text{and} \quad \frac{\partial}{\partial \mathbf{n}} \circ \mathcal{S}_k\left(\frac{\partial u}{\partial \mathbf{n}}\right) + \frac{\partial u}{\partial \mathbf{n}} = \frac{\partial u^i}{\partial \mathbf{n}}$$

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$$S_k\left(\frac{\partial u}{\partial \mathbf{n}}\right) = \gamma(u^I), \quad \text{and} \quad \left(\frac{I}{2} + K_k'\right)\left(\frac{\partial u}{\partial \mathbf{n}}\right) = \frac{\partial u^i}{\partial \mathbf{n}}$$

Direct formulations

- Dirichlet problem (sound-soft problem):

$$A'_{k,\eta} := \frac{1}{2} I_{\mathrm{d}} + K'_k - i\eta S_k, \quad A'_{k,\eta} : L^2(\Gamma) \to L^2(\Gamma)$$

$$A'_{k,\eta} \frac{\partial u}{\partial n} = \frac{\partial u^I}{\partial n} - i\eta\gamma u^I$$

- Neumann problem (sound-hard problem):

$$B_{k,\eta} := H_k + i\eta \left(\frac{1}{2} I_{\mathrm{d}} - K_k \right), \quad B_{k,\eta} : L^2(\Gamma) \to H^{-1}(\Gamma)$$

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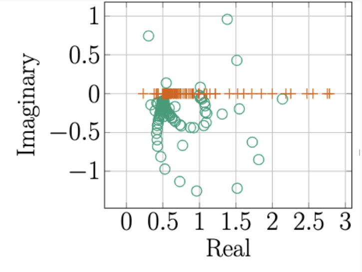
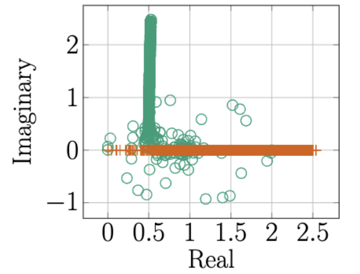
Both are well-posed if $\mathrm{Re}(\eta) \neq 0$, we can use regularization for $B_{k,\eta}$



(a) $A'_{k,k}$



(b) $B_{k,k}$

Figure 2: Spectra for the elliptic cavity

Literature on the sound-soft problem

Papers on $A'_{k,\eta}$:

High-frequency bounds on norms:

- Kress and Spassov 1983; Kress 1985
- Chandler-Wilde and Monk 2008; Chandler-Wilde, Graham, Langdon, and Lindner 2009; Betcke, Chandler-Wilde, Graham, Langdon, and Lindner 2010
- Han and Tacy 2015; Spence, Kamotski, and Smyshlyaev 2015; Baskin, Spence, and Wunsch 2016; J. Galkowski and Spence 2019; Chandler-Wilde, Spence, Gibbs, and Smyshlyaev 2020

Frequency-explicit Galerkin error bounds:

- Banjai and Sauter 2007
- Löhndorf and Melenk 2011; Graham, Löhndorf, Melenk, and Spence 2014
- J. Galkowski, Müller, and Spence 2019

Literature on the sound-hard problem

Only one paper with high-frequency bounds on norms: Boubendir and Turc 2013

For a particular regularised version of $B_{k,\eta}$, they prove

- Sharp bounds on the norm when Ω is a ball,
- Non-sharp bounds on the norm when Ω is smooth,
- Non-sharp bounds on the norm of the inverse when Ω is a ball.

Regularisations for Neumann problems

Idea: Following (Steinbach and Wendland 1998) about first-kind formulation: Use a regularising operator $R : H^{-1}(\Gamma) \to L^2(\Gamma)$

$$B_{k,\eta,R} := RH_k + i\eta \left(\frac{1}{2}I_\mathrm{d} - K_k\right)$$

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Choices for R:

- *Remark*: If $R = i\eta\, \mathsf{P}^+_{\mathrm{NtD}}$, then $B_{k,\eta,R} = i\eta I_d$.
  Approximation of $\mathsf{P}^+_{\mathrm{NtD}}$ (Antoine and Darbas 2021)

- *Calderon relations*: $S_k H_k = -\frac{I}{4} + K_k^2$
    - $R = S_0$ and $\eta \sim 1$
      (Amini and Harris 1990; Anand, Ovall, and Turc 2012)
    - $R = S_{ik}$ and $\eta \sim 1$
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Applications

Why are we interested in k-explicit bounds on $B_{k,\eta,S_{ik}}$ and its inverse?

- Understanding the convergence of iterative solvers (P. M., J. Galkowski, A. Spence, and E. A. Spence 2021, submitted)
  - With strong trapping: $\|(A'_{k_\alpha,k_\alpha})^{-1}\|_{L_2\to L_2} \gtrsim e^{k_\alpha}$ where k_α is a quasi-resonance. What about $B_{k,\eta,S_{ik}}$?
  - Bound on the norm gives bound on the numerical range/pseudo-spectrum
- Frequency-explicit Galerkin error bounds (Galkowski, M., and Spence 2021a, in preparation)
  - Sufficient conditions for quasioptimality

Why is it challenging?

$B_{k,\eta,R}$ **is harder to analyse than** $B_{k,\eta}$

Expressions of the inverse operators:

$$(B_{k,\eta})^{-1} = \mathsf{P}^{+}_{\mathrm{NtD}} - (I - i\eta\,\mathsf{P}^{+}_{\mathrm{NtD}})\,\mathsf{P}^{-,\eta}_{\mathrm{ItD}}$$
$$(B_{k,\eta,R})^{-1} = \mathsf{P}^{+}_{\mathrm{NtD}}\,R^{-1} - (I - i\eta\,\mathsf{P}^{+}_{\mathrm{NtD}}\,R^{-1})\,\mathsf{P}^{-,\eta,\mathsf{R}}_{\mathrm{ItD}}$$

where $\mathsf{P}^{+}_{\mathrm{NtD}}$ is the exterior Neumann-to-Dirichlet operator and $\mathsf{P}^{-,\eta,\mathsf{R}}_{\mathrm{ItD}}$ takes $g \mapsto \gamma^{-}u$ with

$$\Delta u + k^2 u = 0 \text{ in } \Omega \quad \text{and} \quad R\partial_n^{-} u - i\eta\gamma^{-} u = g \text{ on } \Gamma.$$

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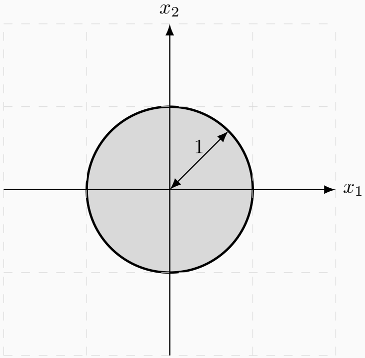
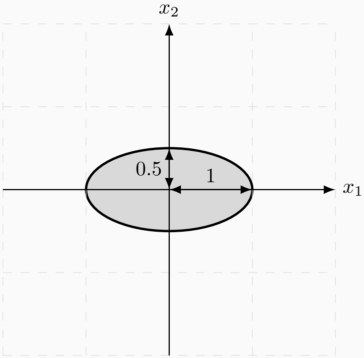
Bounds on the norm



(a) Circle



(b) Ellipse

Figure 3: C^∞ and curved geometries

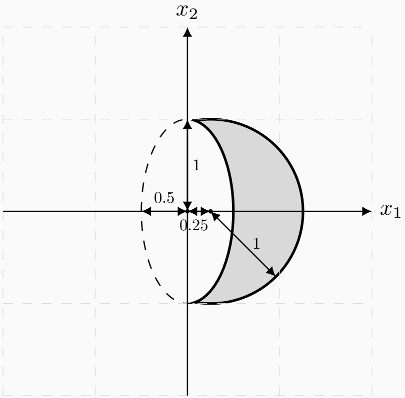
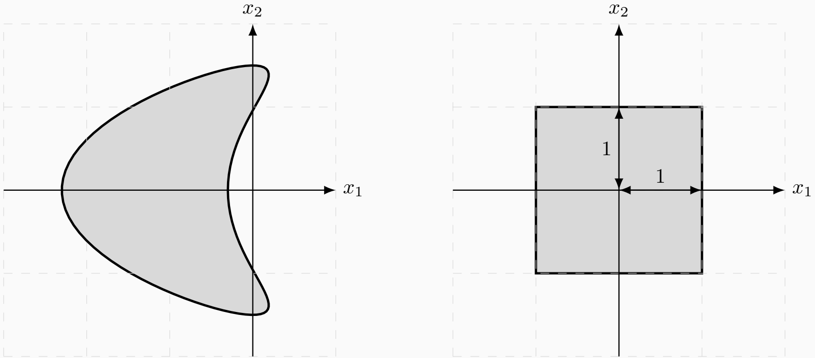


Figure 4: Moon

Figure 5: Piecewise curved geometries



(a) Kite (b) Square

Figure 6: Piecewise smooth geometries

Bounds on the norm

Theorem (Bounds on $\|B_{k,\eta,S_{ik}}\|_{L^2(\Gamma)\to L^2(\Gamma)}$)

(i) If Γ is C^∞ and curved, then

$$\|B_{k,\eta,S_{ik}}\|_{L^2(\Gamma)\to L^2(\Gamma)} \lesssim 1 + |\eta|$$

(ii) If Γ is piecewise curved, then

$$\|B_{k,\eta,S_{ik}}\|_{L^2(\Gamma)\to L^2(\Gamma)} \lesssim |\eta|(1 + k^{1/6}\log(k+2) + (\log(k+2))^{3/2})$$

(iii) If Γ is piecewise smooth, then

$$\|B_{k,\eta,S_{ik}}\|_{L^2(\Gamma)\to L^2(\Gamma)} \lesssim |\eta|(1 + k^{1/4}\log(k+2) + (\log(k+2))^{3/2})$$

Bounds on the norm

Skech of proof:

- Triangle inequality $\implies$

$$\|B_{k,\eta,S_{ik}}\|_{L^2(\Gamma)\to L^2(\Gamma)} \le$$
$$|\eta|\big(1/2 + \|K_k\|_{L^2(\Gamma)\to L^2(\Gamma)}\big) + \|S_{ik}\|_{H_k^{-1}(\Gamma)\to L^2(\Gamma)}\|H_k\|_{L^2(\Gamma)\to H_k^{-1}(\Gamma)}.$$

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- Bounds on $\|K_k\|_{L^2(\Gamma)\to L^2(\Gamma)}$ are known and sharp up to $\log(k)$ (Han and Tacy 2015; J. Galkowski and Spence 2019)
- Using semiclassical analysis,

$$\|S_{ik}\|_{H_k^{-1}(\Gamma)\to L^2(\Gamma)} \lesssim k^{-1}(\log(k+2))^{1/2} \quad \text{and} \quad \|H_k\|_{L^2(\Gamma)\to H_k^{-1}(\Gamma)} \lesssim k\log(k+2)$$

Well-posedness

Theorem (Invertibility on $L^2(\Gamma)$)

If Γ is C^1 and $\eta \in \mathbb{R} \setminus \{0\}$, then $B_{k,\eta,S_{ik}}$ is invertible

Sketch of proof

- *Uniqueness*: Suppose $\phi \in H^{-1/2}(\Gamma)$ such that $B_{k,\eta,S_{ik}}\phi = 0$. Let $u = \mathcal{D}_k\phi$, jump relations imply that

$$\gamma^{\pm}u = \left(\pm\frac{1}{2}I + K_k\right)\phi, \quad \partial_n^{\pm}u = H_k\phi,$$

  - $S_{ik}\partial_n^- u - i\eta\gamma^- u = 0 \implies u = 0$ in Ω.
  - Thus, $\partial^- u = 0$ and by continuity $\partial^+ u = 0$.
  - By wellposedness of the Neumann exterior problem, $u = 0$ in $\mathbb{R} \setminus \overline{\Omega}$ and $\phi = \gamma^+ u - \gamma^- u = 0$.

Well-posedness

- *Identity+compact*: We use the Calderon relations

$$S_k H_k = -\frac{I}{4} + (K_k)^2.$$

We deduce that

$$B_{k,\eta,S_{ik}} = \left(\frac{i\eta}{2} - \frac{1}{4}\right) I + L_{k,\eta},$$

with

$$L_{k,\eta} = (S_{ik} - S_k)H_k - i\eta K_k + (K_k)^2.$$

Then, we use the following mapping properties:

  - K_k is compact on $L^2(\Gamma)$,
  - $H_k : L^2(\Gamma) \to H^{-1}(\Gamma)$,
  - $S_{ik} - S_k : H^{-1}(\Gamma) \to H^1(\Gamma)$

Bounds on the norm of the inverse

Upper bounds on the norm of the inverse

Theorem (Upper bounds on $\|B^{-1}_{k,\eta,S_{ik}}\|_{L^2(\Gamma)\to L^2(\Gamma)}$)

Assume $\eta \in \mathbb{R}\setminus\{0\}$ and is independent of k

(i) If Γ is C^∞ and curved, then

$$\|B^{-1}_{k,\eta,S_{ik}}\|_{L^2(\Gamma)\to L^2(\Gamma)} \lesssim k^{1/3}$$

(ii) If Γ is C^∞ and nontrapping

$$\|B^{-1}_{k,\eta,S_{ik}}\|_{L^2(\Gamma)\to L^2(\Gamma)} \lesssim k^{2/3}$$

(iii) If Γ is C^∞ then there exists $k_0 > 0$ such that given $\delta > 0$ there exists a set $J \subset [k_0,\infty)$ with $|J| \le \delta$ such that, given $\epsilon > 0$, there exists $C = C(k_0,\delta,\epsilon) > 0$ such that, for all $k \in [k_0,\infty)\setminus J$,

$$\|B^{-1}_{k,\eta,S_{ik}}\|_{L^2(\Gamma)\to L^2(\Gamma)} \lesssim Ck^{5d/2+1+\epsilon}$$

(iv) If Γ is C^∞ then there exists $k_0 > 0$, $\alpha > 0$, and $C > 0$ such that, for all $k \ge k_0$, $\|B^{-1}_{k,\eta,S_{ik}}\|_{L^2(\Gamma)\to L^2(\Gamma)} \lesssim C\exp(\alpha k)$

Lemma (Baskin, Spence, and Wunsch 2016)

$$(B_{k,\eta,R})^{-1} = \mathsf{P}^{+}_{\mathrm{NtD}}\, S_{ik}^{-1} - (I - i\eta\, \mathsf{P}^{+}_{\mathrm{NtD}}\, S_{ik}^{-1})\, \mathsf{P}^{-,\eta,\mathsf{S}_{ik}}_{\mathrm{ItD}}$$

Sketch of proof

Given g and ϕ satisfying $B_{k,\eta,S_{ik}}\phi = g$, let $u = \mathcal{D}_k\phi$. Jump relations imply

$$S_{ik}\partial_n^- u - i\eta\gamma^- u = g \Leftrightarrow \gamma^- u = \mathsf{P}^{-,\eta,\mathsf{S}_{ik}}_{\mathrm{ItD}}\, g,$$

and

$$\begin{aligned} \phi = \gamma^+ u - \gamma^- u &= \mathsf{P}^{+}_{\mathrm{NtD}}(\partial_n^+ u) - \mathsf{P}^{-,\eta,\mathsf{S}_{ik}}_{\mathrm{ItD}}\, g \\ &= \mathsf{P}^{+}_{\mathrm{NtD}}\, S_{ik}^{-1}(g + i\eta\gamma^- u) - \mathsf{P}^{-,\eta,\mathsf{S}_{ik}}_{\mathrm{ItD}}\, g \end{aligned}$$

Bounds on the norm of the inverse

We deduce

$$\|(B_{k,\eta,R})^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)} \leq \| \mathsf{P}_{\mathrm{ItD}}^{-,\eta,\mathsf{S}_{\mathsf{ik}}} \|_{L^2(\Gamma)\to L^2(\Gamma)} + (1+|\eta|\| \mathsf{P}_{\mathrm{ItD}}^{-,\eta,\mathsf{S}_{\mathsf{ik}}} \|_{L^2(\Gamma)\to L^2(\Gamma)})\|\mathsf{P}_{\mathrm{NtD}}^{+}\|_{H_k^{-1}(\Gamma)\to L^2(\Gamma)}\|S_{ik}^{-1}\|_{L^2(\Gamma)\to H_k^{-1}(\Gamma)}.$$

- We prove bounds on $\|\mathsf{P}_{\mathrm{NtD}}^{+}\|_{L^2(\Gamma)\to H_k^1(\Gamma)}$ using results from Baskin, Spence, and Wunsch 2016; Lafontaine, Spence, and Wunsch 2020; Burq 1998.
- We show existence of $\mathsf{P}_{\mathrm{ItD}}^{-,\eta,\mathsf{S}_{\mathsf{ik}}}$ and $\|\mathsf{P}_{\mathrm{ItD}}^{-,\eta,\mathsf{S}_{\mathsf{ik}}}\|_{L^2(\Gamma)\to L^2(\Gamma)} \lesssim 1$

Interior impedance-to-Dirichlet map

Lemma (Bound of $\mathsf{P}_{\mathrm{ItD}}^{-,\eta,\mathsf{S}_{ik}} : L^2(\Gamma) \to L^2(\Gamma)$)

If Γ is C^∞ and $\eta \in \mathbb{R} \setminus \{0\}$, there exists k_0 and $C > 0$ (independent of k) such that, for all $k \geq k_0$

$$\|\mathsf{P}_{\mathrm{ItD}}^{-,\eta,\mathsf{S}_{ik}}\|_{L^2(\Gamma)\to L^2(\Gamma)} \leq C.$$

Sketch of proof

- $\|\mathsf{P}_{\mathrm{ItD}}^{-,\eta,\mathsf{S}_{ik}}\|_{H^1_k(\Gamma)\to H^1_k(\Gamma)} \leq C$ (Galkowski, Lafontaine, and Spence 2021)
- $\|\mathsf{P}_{\mathrm{ItD}}^{-,\eta,\mathsf{S}_{ik}}\|_{L^2(\Gamma)\to L^2(\Gamma)} \sim \|\mathsf{P}_{\mathrm{ItD}}^{-,\eta,\mathsf{S}_{ik}}\|_{H^1_k(\Gamma)\to H^1_k(\Gamma)}$

Interior impedance-to-Dirichlet map

Theorem (Existence of $\mathsf{P}_{\mathrm{ItD}}^{-,\eta,\mathsf{S}_{ik}} : H^{1/2}(\Gamma) \to H^{1/2}(\Gamma)$)

Let Γ be Lipschitz, $g \in H^{1/2}(\Gamma)$ and $\eta \in \mathbb{R} \setminus \{0\}$, there exists a unique solution $u \in H^1(\Omega)$ to

$$\Delta u + k^2 u = 0 \text{ in } \Omega \quad \text{and} \quad S_{ik}\partial_n^- u - i\eta\gamma^- u = g \text{ on } \Gamma,$$

and thus $\mathsf{P}_{\mathrm{ItD}}^{-,\eta,\mathsf{S}_{ik}} : H^{1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is well-defined.

Sketch of proof

We introduce the variational formulation: find $u \in H^1(\Omega)$ such that $a(u,v) = F(v)$ for all $v \in H^1(\Omega)$ with

$$a(u,v) := \int_\Omega \left(\nabla u \cdot \overline{\nabla v} - k^2 u\, \overline{v}\right) - i\eta\left\langle (S_{ik})^{-1}\gamma^- u, \gamma^- v\right\rangle_\Gamma$$
$$F(v) := \left\langle (S_{ik})^{-1} g, \gamma^- v\right\rangle_\Gamma.$$

Sketch of proof

- *Uniqueness*: If $g = 0$

$$0 = |\mathrm{Im}(a(u,u))| = |\eta \, \mathrm{Re}(\langle (S_{ik})^{-1}\gamma^- u, \gamma^- u\rangle_\Gamma)| \gtrsim \|(S_{ik})^{-1}\gamma^- u\|_{H^{-1/2}(\Gamma)}$$

- *Existence*:

$$\mathrm{Re}(a(v,v)) = \|\nabla v\|^2_{L^2(\Omega)} - k^2\|v\|^2_{L^2(\Omega)}$$

By Fredholm theory, the solution exists and is unique.

Interior impedance-to-Dirichlet map

Lemma

Assume Γ *is* C^∞, $\eta \in \mathbb{R} \setminus \{0\}$,

- $\mathsf{P}^{\text{-},\eta,\mathsf{S_{ik}}}_{\mathrm{ItD}}$ *has a unique extension to a bounded operator* $H^1(\Gamma) \to H^1(\Gamma)$.
- $\|S_{ik}^{-1}\,\mathsf{P}^{\text{-},\eta,\mathsf{S_{ik}}}_{\mathrm{ItD}}\|_{L^2(\Gamma)\to H_k^{-1}(\Gamma)} = \|S_{ik}^{-1}\,\mathsf{P}^{\text{-},\eta,\mathsf{S_{ik}}}_{\mathrm{ItD}}\|_{H_k^1(\Gamma)\to L^2(\Gamma)}$

Sketch of proof

- $\mathsf{P}^{\text{-},\eta,\mathsf{S_{ik}}}_{\mathrm{ItD}} = -S_{ik}\big(B_{\eta,k,S_{ik}}\big)^{-1}\left(\tfrac{1}{2}I - K'_k\right)S_{ik}^{-1}$ and mapping properties
- $\left\langle S_{ik}^{-1}\,\mathsf{P}^{\text{-},\eta,\mathsf{S_{ik}}}_{\mathrm{ItD}}\,\phi,\,\psi\right\rangle_{\Gamma,\mathbb{R}} = \left\langle \phi,\,S_{ik}^{-1}\,\mathsf{P}^{\text{-},\eta,\mathsf{S_{ik}}}_{\mathrm{ItD}}\,\psi\right\rangle_{\Gamma,\mathbb{R}}$ for all $\phi,\psi \in H^{1/2}(\Gamma)$

$$\gamma^- u = \mathsf{P}^{\text{-},\eta,\mathsf{S_{ik}}}_{\mathrm{ItD}}\,\phi \quad \text{and} \quad \gamma^- v = \mathsf{P}^{\text{-},\eta,\mathsf{S_{ik}}}_{\mathrm{ItD}}\,\psi.$$

$$\left\langle \gamma^- u, \partial_n^- v\right\rangle_{\Gamma,\mathbb{R}} - \langle \gamma^- v, \partial_n^- u\rangle_{\Gamma,\mathbb{R}} = \int_\Omega u\Delta v - v\Delta u = 0,$$

Interior impedance-to-Dirichlet map

Corollary

If Γ *is* C^∞, $k>0$, *and* $\operatorname{Re}\eta \neq 0$, *then*

$$\|\mathsf{P}^{-,\eta,\mathsf{S}_{ik}}_{\mathrm{ItD}}\|_{L^2(\Gamma)} \sim \|\mathsf{P}^{-,\eta,\mathsf{R}'}_{\mathrm{ItD}}\|_{H^1_k(\Gamma)\to H^1_k(\Gamma)}.$$

Sketch of proof

$$k\|\mathsf{P}^{-,\eta,\mathsf{S}_{ik}}_{\mathrm{ItD}}\|_{H^1_k(\Gamma)\to H^1_k(\Gamma)} \lesssim \|S_{ik}^{-1}\,\mathsf{P}^{-,\eta,\mathsf{S}_{ik}}_{\mathrm{ItD}}\|_{H^1_k(\Gamma)\to {}^2(\Gamma)} \lesssim k\|\mathsf{P}^{-,\eta,\mathsf{S}_{ik}}_{\mathrm{ItD}}\|_{H^1_k(\Gamma)\to H^1_k(\Gamma)}$$

and

$$k\|\mathsf{P}^{-,\eta,\mathsf{S}_{ik}}_{\mathrm{ItD}}\|_{L^2(\Gamma)\to L^2(\Gamma)} \lesssim \|S_{ik}^{-1}\,\mathsf{P}^{-,\eta,\mathsf{S}_{ik}}_{\mathrm{ItD}}\|_{L^2(\Gamma)\to H^{-1}_k(\Gamma)} \lesssim k\|\,\mathsf{P}^{-,\eta,\mathsf{S}_{ik}}_{\mathrm{ItD}}\|_{L^2(\Gamma)\to L^2(\Gamma)}.$$

Lower bounds on the norm of the inverse

Definition

A family of Neumann quasimodes of quality $\epsilon(k)$ is a sequence $\{(u_j, k_j)\}_{j=1}^{\infty} \subset H^2_{\mathrm{loc}}(\mathbb{R}^d \setminus \overline{\Omega}) \times \mathbb{R}$ with $\partial_n^+ u = 0$ on Γ such that the frequencies $k_j \to \infty$ as $j \to \infty$ and there exists a compact subset $\mathcal{K} \subset \mathbb{R}^d \setminus \overline{\Omega}$ such that, for all j, $\operatorname{supp} u_j \subset \mathcal{K}$,

$$\|(\Delta + k_j^2)u_j\|_{L^2(\mathbb{R}^d\setminus\overline{\Omega})} \leq \epsilon(k_j), \quad \text{and} \quad \|u_j\|_{L^2(\mathbb{R}^d\setminus\overline{\Omega})} = 1.$$

Theorem (Lower bounds on $\|(B_{k,\eta,S_{ik}})^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)}$)

If there exists a family of Neumann quasimodes with quality $\epsilon(k)$, then there exists $C > 0$ (independent of j) such that

$$\|(B_{k_j,\eta,R})^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)} \geq C\left(\frac{1}{\epsilon(k_j)} - \frac{1}{k_j}\right) k_j^{1/2}\Big(\|S_{ik}\|_{L^2(\Gamma)\to L^2(\Gamma)}k_j + |\eta|\Big)^{-1}.$$

Sketck of proof

- With $\Delta u_j + k_j^2 u_j = g_j$, we have $\|g_j\|_{L^2(\mathbb{R}^d\setminus\overline{\Omega})} \leq \epsilon(k_j)$.
- $u_j^I(x) := \mathcal{R}_k g_j(x) = \int_{\mathbb{R}^d\setminus\overline{\Omega}} \Phi_{k_j}(x,y) g_j(y)\,\mathrm{d}y$ satisfies $\Delta u_j^I + k_j^2 u_j^I = g_j$
- $u_j^S = u_j - u_j^I$ is solution of a Neumann exterior problem,

$$-(\mathcal{S}_{k_j}\partial_n u_j^S)(x) + (\mathcal{D}_{k_j} u_j^S)(x) = \begin{cases} u_j^S & \text{for } x \in \mathbb{R}^d \setminus \overline{\Omega}, \\ 0 & \text{for } x \in \Omega, \end{cases}$$

$$-(\mathcal{S}_{k_j}\partial_n u_j^I)(x) + (\mathcal{D}_{k_j} u_j^I)(x) = \begin{cases} 0 & \text{for } x \in \mathbb{R}^d \setminus \overline{\Omega}, \\ u_j^I & \text{for } x \in \Omega. \end{cases}$$

$$\implies u_j = u_j^I + \mathcal{D}_{k_j}\gamma^+ u_j \quad \text{in } \mathbb{R}^d \setminus \overline{\Omega},$$
$$\implies B_{k,\eta,S_{ik}}(\gamma^+ u_j) = f_j, \quad \text{where} \quad f_j := -\big(S_{ik}\partial_n^+ - i\eta\gamma^+\big)u^I.$$

Numerical experiments

Numerical setting

- *Discretisation*: BEM with continuous piecewise-linear basis functions, 10 points by wavelength
- *Software*: FreeFEM and its interfaces to BemTool[1], HTool[2], SLEPc and ScaLAPACK.

We solve singular value problems of $\mathsf{M}^{-1}\mathsf{B}_{k,\eta,S_{ik}}$ for $k = 5 \times 2^n$ with $n = 0, 1, \dots, 8$, i.e., $k \in (5, 1280)$.

[1] `https://github.com/xclaeys/BemTool`
[2] `https://github.com/htool-ddm/htool`

	Circle	Ellipse	Expected
$\lVert B_{k,\eta,S_{ik}} \rVert$	~ 1	~ 1	$\lesssim 1$
$\lVert (B_{k,\eta,S_{ik}})^{-1} \rVert$	$\sim k^{0.34}$	$\sim k^{0.28}$	$\lesssim k^{1/3}$

Table 1: Comparison of the k-dependence of the computed norms for the circle and the ellipse.

Remark

We also have $\mathrm{cond}(A'_{k,k}) \sim k^{1/3}$.

Piecewise curved geometries

	Moon	Expected
$\lVert B_{k,\eta,S_{ik}} \rVert$	$\sim k^{0.15}$	$\lesssim k^{1/6}\log(k+2)$
$\lVert (B_{k,\eta,S_{ik}})^{-1} \rVert$	$\sim k^{0.41}$	$\lesssim k^{2/3}$

Table 2: Comparison of the k-dependence of the computed norms for the moon obstacle.

Remark

Note that the bound on $(B_{k,\eta,S_{ik}})^{-1}$ *does not apply since* Γ *is not* C^∞.

Piecewise smooth geometries

	Kite	Square	Expected
$\|B_{k,\eta,S_{ik}}\|$	$\sim k^{0.21}$	$\sim k^{0.16}$	$\lesssim k^{1/4}\log(k+2)$
$\|(B_{k,\eta,S_{ik}})^{-1}\|$	$\sim k^{0.41}$	$\sim k^{0.13}$	$\lesssim k^{2/3}$

Table 3: Comparison of the k-dependence of the computed norms for the kite and the square.

Remark

Note that the bound on $(B_{k,\eta,S_{ik}})^{-1}$ does not apply to the square since Γ is not C^∞.

Conclusion

Final remarks: In Galkowski, M., and Spence (2021b). *High-frequency estimates on boundary integral operators for the Helmholtz exterior Neumann problem*. arXiv: `2109.06017`

- We consider a slightly more general regularising operator
- All the previous results also extend to $B'_{k,\eta,S_{ik}}$

Outlook

- Frequency-explicit Galerkin error bounds for h-BEM with $B_{k,\eta,S_{ik}}$ (and $A'_{k,\eta}$!) (Galkowski, M., and Spence 2021a)

Conclusion

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Outlook

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Thank you for your attention!

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