On the multiplicity of the second eigenvalue of the Laplacian in non simply connected domains.

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(after Helffer, Hoffmann-Ostenhof, Jauberteau, Léna)

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The motivating problem is to analyze the multiplicity of the k-th eigenvalue of the Dirichlet problem in a domain Ω in $\mathbb{R}^2$.
It is for example an old result of Cheng [Ch1976], that the multiplicity of the second eigenvalue is at most 3 in the case of a compact manifold.
In [HOHON1997b] (Hoffmann-Ostenhof (M+T) and N. Nadirashvili) an example with multiplicity 3 is proposed as a side product of the production of a counter example to the nodal line conjecture (see also [HOHON1997a], and the papers by Fournais [Fo2001] and Kennedy [Ke2013] who extend to higher dimensions these counter examples, introducing new methods).

This example is based on the spectral analysis of the Laplacian in domains consisting of a disc in which we have introduced on an interior concentric circle suitable cracks.
We discuss the initial proof and complete it by one missing argument.

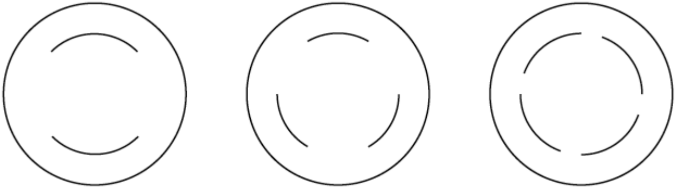

Figure: The domains with cracks for $N = 2$, $N = 3$ and $N = 4$.

Numerics and theory

Although not needed for the positive results, we present numerical results illustrating why some argument has to be modified and propose a fine theoretical analysis of the spectral problem when the cracks are closed.

Nodal line conjecture

This celebrated conjecture by Payne (1967) says that the nodal set of the second eigenfunction of the Dirichlet Laplacian in $\Omega \subset \mathbb{R}^2$ can not be a closed curve in Ω. Another formulation is that the nodal set consists of a line joining two points of the boundary. It has been proved to hold in the convex case (Melas, Alessandrini).

The first counterexample was given by (Hoffmann-Ostenhof (M+T)-Nadirashvili) and the starting point is the introduction of two concentric open discs B_{R_1} and B_{R_2} with $0 < R_1 < R_2$ and the corresponding annulus $M_{R_1,R_2} = B_{R_2} \setminus \bar{B}_{R_1}$. We choose R_1 and R_2 such that

$$\lambda_1(B_{R_1}) < \lambda_1(M_{R_1,R_2}) < \lambda_2(B_{R_1}) \,, \tag{1}$$

where, for $\omega \subset \mathbb{R}^2$ bounded, $\lambda_j(\omega)$ denotes the j-th eigenvalue of the Dirichlet Laplacian H in ω.

For fixed R_1, $\lambda_1(M_{R_1,R_2})$ tends to $+\infty$ as $R_2 \to R_1$ (from above) and tends to 0 as $R_2 \to +\infty$. Moreover $R_2 \mapsto \lambda_1(M_{R_1,R_2})$ is decreasing.

Assumptions

We introduce

$$D_{R_1,R_2} = B_{R_1} \cup M_{R_1,R_2}$$

and observe that under above condition

$$\begin{aligned} \lambda_1(D_{R_1,R_2}) &= \lambda_1(B_{R_1}) \\ \lambda_2(D_{R_1,R_2}) &= \lambda_1(M_{R_1,R_2}) \\ \lambda_3(D_{R_1,R_2}) &= \min(\lambda_2(B_{R_1}), \lambda_2(M_{R_1,R_2}))\,. \end{aligned} \tag{2}$$

If Condition (1) is important in the construction of the counter-example to the nodal line conjecture, the weaker assumption

$$\max(\lambda_1(B_{R_1}), \lambda_1(M_{R_1,R_2})) < \min(\lambda_2(B_{R_1}), \lambda_2(M_{R_1,R_2}))\,. \tag{3}$$

suffices for the multiplicity question. Under this condition, we have :

$$\begin{aligned} \lambda_1(D_{R_1,R_2}) &= \min(\lambda_1(B_{R_1}), \lambda_1(M_{R_1,R_2})) \\ \lambda_2(D_{R_1,R_2}) &= \max(\lambda_1(B_{R_1}), \lambda_1(M_{R_1,R_2})) \\ \lambda_3(D_{R_1,R_2}) &= \min(\lambda_2(B_{R_1}), \lambda_2(M_{R_1,R_2}))\,, \end{aligned} \tag{4}$$

and it is not excluded (we are in the non connected situation) to consider the case $\lambda_1(D_{R_1,R_2}) = \lambda_2(D_{R_1,R_2})$.

We now carve holes in ∂B_{R_1} such that D_{R_1,R_2} becomes a domain. For $N \in \mathbb{N}^*$ and $\epsilon \in [0, \frac{\pi}{N}]$, we introduce (see Figure 1 for $N = 2, 3, 4$)

$$\mathfrak{D}(N,\epsilon) = D_{R_1,R_2} \cup_{j=0}^{N-1} \{x \in \mathbb{R}^2\,,\; r = R_1\,,\; \theta \in (\frac{2\pi j}{N} - \epsilon, \frac{2\pi j}{N} + \epsilon)\}\,. \quad (5)$$

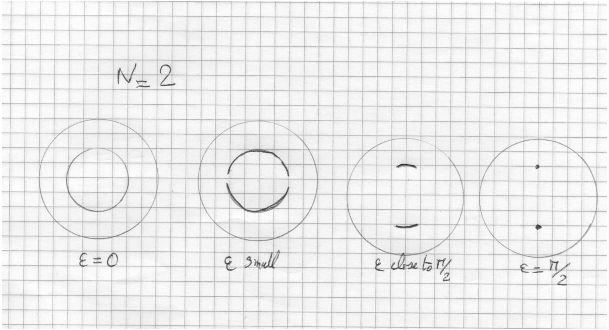

Figure: The domains with cracks for $N = 2$.

The theorem stated in [HOHON2] is the following :

Main Theorem

Let $N \geq 3$, then there exists $\epsilon \in (0, \frac{\pi}{N})$ such that $\lambda_2(\mathfrak{D}(N,\epsilon))$ has multiplicity 3.

The proof given in [HOHON2] works only for even integers $N \geq 4$. So we improve this result by giving a proof for $N \geq 3$, therefore giving the first example of an open set $\Omega := \mathfrak{D}(3,\epsilon)$ where the number of components of $\partial\Omega$ equals 4 .

It is natural to expect that for other domains with the appropriate symmetry and the same connectivity as in the Main Theorem one can also show using a similar approach as presented in this work that the second eigenvalue can have multiplicity 3.

The main theorem leads to the following question :
Is there a bounded domain $\Omega \subset \mathbb{R}^2$ whose boundary $\partial\Omega$ has strictly less than 4 components so that $\lambda_2(\Omega)$ has multiplicity 3 ?

This is also a motivation for analyzing the cases $N = 1, 2$.

The natural conjecture would be that for simply connected domains Ω, $\lambda_2(\Omega)$ has at most multiplicity 2. The convex case is known (Lin).

N for the nodal line conjecture

Till recently the N leading to an example with a closed nodal line was estimated to 10^4 !

Recently (March 2021), in a paper submitted to ArXiv, J. Dahne, J. Gomez-Serrano and K. Hou have produced an example with six holes. Instead of considering a carved circle they start from an hexagone and the holes are given by six equilateral triangles.

About numerics for the multiplicity question

We consider for the numerics the following specific choice of the pair (R_1, R_2). We take $R_2 = 1$, and then take as $0 < R_1 < 1$ the radius of the circle on which the second radial eigenfunction vanishes. This corresponds to $R_1 \sim 0.4356$.

In this case, we have $\lambda_1(B_{R_1}) = \lambda_1(M_{R_1,R_2})$ and the interesting point is that $\lambda_1(B_{R_1}) = \lambda_6(B_{R_2})$ is an eigenvalue of the Dirichlet Laplacian in $\mathfrak{D}(N, \epsilon)$ for any $\epsilon \in [0, \frac{\pi}{N}]$.

Figure 2 : N=3

We can predict as $N = 3$ a second eigenvalue of multiplicity 3 for $\epsilon \sim 0.29$. A second crossing appears for $\epsilon \sim 0.96$ but corresponds to a third eigenvalue of multiplicity 3. The eigenvalues correspond to $\ell = 0$ (invariant case) and to $\ell = 1$ (other symmetry space), the eigenvalues for $\ell = 1$ having multiplicity 2.

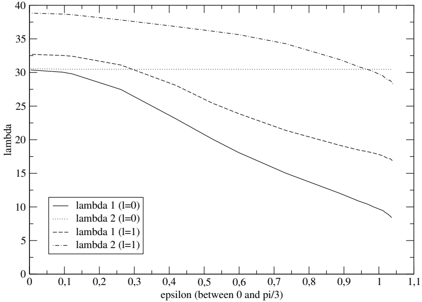

Figure: Six lowest eigenvalues in function of $\epsilon \in (0, \frac{\pi}{3})$.

The case $N = 4$

We see a first crossing for $\epsilon \sim 0.27$ where the multiplicity becomes 3.
Two other crossings occur for $\epsilon \sim 0.475$ and $\epsilon \sim 0.782$.
The eigenvalues correspond to $\ell = 0\,,\,1\,,\,2$.
The eigenvalues for $\ell = 1$ having multiplicity 2.
The eigenvalues for $\ell = 0$ and 2 are simple for $\epsilon \in (0, \frac{\pi}{4})$

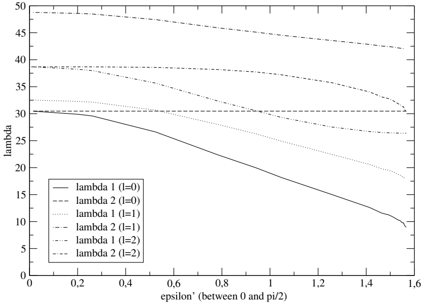

Figure: 8 lowest eigenvalues in $\mathfrak{D}(N, \epsilon'/2))$ in function of $\epsilon' = 2\epsilon \in (0, \frac{\pi}{2})$.

Symmetry spaces (explaining the "ℓ")

We recall some basic representation theory for finite groups.

We consider a Hamiltonian which is the Dirichlet realization of the Laplacian in an open set Ω which is invariant by the action of the group G_N generated by the rotation g by $\frac{2\pi}{N}$.

The initial Hilbert space is $\mathcal{H} := L^2(\Omega, \mathbb{R})$ but it is also convenient to work in $\mathcal{H}_{\mathbb{C}} := L^2(\Omega, \mathbb{C})$.
In this case, it is natural to analyze the eigenspaces attached to the irreducible representations of the group G_N.

The theory will in particular apply for the family of open sets $\Omega = \mathfrak{D}(N, \epsilon)$.

The theory is simpler for complex Hilbert spaces i.e. $\mathcal{H}_{\mathbb{C}} := L^2(\Omega, \mathbb{C})$, but the multiplicity property appears when considering operators on real Hilbert spaces, i.e $\mathcal{H} := L^2(\Omega, \mathbb{R})$.

In $\mathcal{H}_{\mathbb{C}}$, we introduce for $\ell = 0, \cdots, N-1$,

$$\mathcal{B}_\ell = \{w \in \mathcal{H}_{\mathbb{C}} \mid gw = e^{2\pi i \ell / N} w\}\,. \tag{6}$$

For $\ell = 0$, this corresponds to the invariant situation. Hence in the model above (where $\Omega = B_{R_2}$) u_0 and u_6 belong to $\mathcal{B}_0$.

We also observe that the complex conjugation sends $\mathcal{B}_\ell$ onto $\mathcal{B}_{N-\ell}$.

Hence, except in the cases $\ell = 0$ and $\ell = \frac{N}{2}$ the corresponding eigenspaces are of even dimension.

The second case appears only if N is even.

For $2\ell \neq N$, one can alternately come back to real spaces by introducing for $0 < \ell < \frac{N}{2}$ ($\ell \in \mathbb{N}$)

$$\mathcal{C}_\ell = \mathcal{B}_\ell \oplus \mathcal{B}_{N-\ell} \tag{7}$$

and observing that $\mathcal{C}_\ell$ can be recognized as the complexification $\mathcal{A}_\ell \otimes \mathbb{C}$ of the real space $\mathcal{A}_\ell$

$$\mathcal{A}_\ell = \{u \in \mathcal{H} \mid u - 2\cos(2\ell\pi/N)gu + g^2u = 0\}\,. \tag{8}$$

For $\ell = 0$ and $\ell = \frac{N}{2}$ (if N is even), we define $\mathcal{A}_\ell$ by

$$\mathcal{B}_\ell = \mathcal{A}_\ell \otimes \mathbb{C}\,. \tag{9}$$

Under the invariance condition on the domain, the Dirichlet Laplacian commutes with the natural action of g in L^2. Hence we get for $0 \leq \ell \leq N/2$ a family of well defined selfadjoint operators $H^{(\ell)}$ obtained by restriction of H to $\mathcal{A}_\ell$.

Note that except for $\ell = 0$ and $\ell = \frac{N}{2}$ all the eigenspaces of $H^{(\ell)}$ have even multiplicity.

The other point is that we have continuity and monotonicity with respect to ϵ of the eigenvalues (see Stollman).

Note also that

$$\sigma(H(\epsilon, N)) = \cup_{0 \leq \ell \leq \frac{N}{2}} \sigma(H^{(\ell)}(\epsilon, N))\,.$$

Remark

When N is even, a particular role is played by $g^{\frac{N}{2}}$ which corresponds to the inversion considered in [HOHON1997b].

One can indeed decompose the Hilbert space $\mathcal{H}$ (or $\mathcal{H}_{\mathbb{C}}$) using the symmetry with respect to $g^{\frac{N}{2}}$ and get the decomposition

$$\mathcal{H} = \mathcal{H}^{even} \oplus \mathcal{H}^{odd} \,, \tag{10}$$

and

$$H(\epsilon, N) = H^{even}(\epsilon, N) \oplus H^{odd}(\epsilon, N) \,. \tag{11}$$

One can compare this decomposition with the previous one.
We observe that $\mathcal{A}_\ell$ belongs to $\mathcal{H}^{even}$ if ℓ is even and to $\mathcal{H}^{odd}$ if ℓ is odd.

Upper bound : the regularity assumptions in Cheng's statement revisited

In [Ch1976], S.Y. Cheng proved that the multiplicity of the second eigenvalue is at most 3. This bound is sharp, achieved for the round metric on $\mathbb{S}^2$. Cheng's proof is actually using a regularity assumption which is not satisfied by $D(N, \epsilon)$.

This domain has indeed cracks and we need a description of the nodal line structure near corners or cracks.

We recall that for an eigenfunction u the nodal set $N(u)$ of u is defined by

$$\mathcal{N}(u) := \overline{\{x \in \Omega \,, | \, u(x) = 0\}}\,.$$

The analysis in this case is treated in Helffer, Hoffmann-Ostenhof, and Terracini [5] (Theorem 2.6).

With this complementary analysis near the cracks, we can follow the proof of Hoffmann-Ostenhof–Michor-Nadirashvili [8] (Weak theorem).

This proof includes an extended version of Euler's Polyhedral formula

Proposition Euler

Let Ω be a $C^{1,+}$-domain with possibly corners of opening[a] $\alpha\pi$ for $0 < \alpha \leq 2$. If u is an eigenfunction of the Dirichlet Laplacian in Ω, $\mathcal{N}$ denotes the nodal set of u and $\mu(\mathcal{N})$ denotes the cardinality of the components of $\Omega \setminus \mathcal{N}$, i.e. the number of nodal domains, then, if $\mathcal{N}$ is not empty,

$$\mu(\mathcal{N}) \geq \sum_{x \in \mathcal{N} \cap \Omega} (\nu(x) - 1) + 2\,, \tag{12}$$

where $\nu(x)$ is the multiplicity of the critical point $x \in \mathcal{N}$ (i.e. the number of lines crossing at x).

a. $\alpha = 2$ corresponds to the crack case.

Origin of this formula

Proposition HOMiNa

Let Ω be an open set in $\mathbb{R}^2$ with piecewise C^1 boundary and $\mathcal{D}$ be a nodal partition of an eigenfunction u of the Dirichlet Laplacian in Ω with $\mathcal{N} := \overline{\{u^{-1}(0) \cap \Omega\}}$ as boundary set. Let b_0 be the number of components of $\partial\Omega$ and b_1 be the number of components of $\mathcal{N} \cup \partial\Omega$. Denote by $\nu(\mathbf{x}_i)$ and $\rho(\mathbf{y}_i)$ the number of curves crossing at some critical points $\mathbf{x}_i \in X(\mathcal{N})$, respectively some boundary points $\mathbf{y}_i \in Y(\mathcal{N})$. Then

$$k = 1 + b_1 - b_0 + \sum_{\mathbf{x}_i \in X(\mathcal{N})} \Big(\nu(\mathbf{x}_i) - 1\Big) + \frac{1}{2} \sum_{\mathbf{y}_i \in Y(\mathcal{N})} \rho(\mathbf{y}_i)\,. \qquad (13)$$

Here the structure of the nodal set of an eigenfunction plays an important role (Bers (1955), Alessandrini (1987),...).

For a second eigenfunction $\mu(\mathcal{N}) = 2$, (this is a very simple and particular case of Courant's theorem) and the upper bound of the multiplicity by 3 comes by contradiction.

Assuming that the multiplicity of the second eigenvalue is ≥ 4, one can, for any $x \in \Omega$, construct some u_x in the second eigenspace such that u_x has a critical zero at x (hence with $\nu(x) \geq 2$). This is simply by linear algebra (we have three equations).

This gives the contradiction with Euler's formula. Hence we have

Proposition

Let Ω be a $C^{1,+}$-domain with possibly corners of opening $\alpha\pi$ for $0 < \alpha \leq 2$. Then the multiplicity of the second eigenvalue of the Dirichlet Laplacian in Ω is not larger than 3.

Remarks and complements

- An upper bound of the multiplicity by 2 is obtained by C.S. Lin when Ω is convex (see [Li1987]).

  Lin's theorem can be extended to the case of a simply connected domain for which the nodal line conjecture holds.
  If the multiplicity of the second eigenvalue is larger than 2, one can indeed find in the associated spectral space an eigenfunction whose nodal set contains a point in the boundary where two half lines hit the boundary.
  This will contradict either the nodal line conjecture or Courant's theorem. See also [HOHON1997b] for some sufficient conditions on domains for the nodal line conjecture to hold.
- There are no result of this kind in dimension ≥ 3. Yves Colin de Verdière [CdV1987] has for example shown that we can contruct for any N a compact manifold for which the multiplicity of the second eigenvalue is N. According to Nadirashvili (private communication to the second author) this holds also for the Dirichlet problem.

Multiplicity bounds for closed surface of genus 0. (HOHONa1999]

Let (M, g) be a closed connected smooth Riemannian surface with genus 0, i.e. M is topologically a sphere (in particular it is simply-connected with Euler characteristic 2), and g a smooth metric. We are also given a smooth, real valued function V on M, and we consider the eigenvalue problem for the Schrödinger operator $-\Delta + V$ on M, with eigenvalues $(\lambda_k)_{k=1}^{\infty}$. Nadirashvili in [Na1987] proved that, for $k \geq 3$,

$\mathrm{mult}(\lambda_k) \leq (2k-1)$. The general idea of the proof is to investigate the zeros of eigenfunctions u in the eigenspace $U(\lambda_k)$, to use the fact that $\kappa(u) \leq k$ (Courant's theorem), and Euler's formula for the nodal partitions $\mathcal{D}_u$. In [HOHONa1999] , M. and T. Hoffmann-Ostenhof and Nadirashvili, prove the following result.

Theorem

For (M, g, V) as above, and for $k \geq 3$, $\mathrm{mult}(\lambda_k) \leq (2k-3)$.

Comments on HoMiNa (1999) following Bérard-Helffer

- There is a huge litterature on the subject. See a survey in preparation of P. Bérard and B. Helffer.
- Hoffmann-Ostenhof-Michor-Nadirashvili (1999), extending Chen's result, state the following :
  Let Ω be a regular bounded domain in $\mathbb{R}^2$ and consider the Dirichlet eigenvalue problem for the Laplacian or for a Schrödinger operator of the form $-\Delta + V$. Then, for $k \geq 3$, the multiplicity of the k-th Dirichlet eigenvalue of a plane bounded domain is less than or equal to $(2k-3)$.
- Some proofs have to be clarified. In particular, A. Berdnikov (2018) observed a gap in the proof of [HoMiNa] in the non simply connected case. Hence, if the bound $\leq (2k-2)$ seems OK in the non simply connected case, the bound $(2k-3)$ seems open in the non simply connected case. There is at the moment also a gap in the simply connected case.

We first observe that for the disk of radius R we have

$$\lambda_1(B_R) < \lambda_2(B_R) = \lambda_3(B_R) < \lambda_4(B_R) = \lambda_5(B_R) < \lambda_6(B_R)\,. \qquad (14)$$

The eigenfunctions u_1 and u_6 are radial. We will use this property with $R = R_2$.

Proposition

For $N \geq 3$, there exists $\epsilon \in (0, \frac{\pi}{N})$ s.t. $\lambda_2(H(\epsilon, N))$ belongs to $\sigma(H^{(\ell)}(\epsilon, N))$ for some $0 < \ell < \frac{N}{2}$ AND to $\sigma(H^{(\ell)}(\epsilon, N))$ for $\ell = 0$ or (in the case N even) $\frac{N}{2}$.
In particular, the multiplicity of λ_2 for this value of ϵ is exactly 3.

Note that the condition $N \geq 3$ implies the existence of at least one $\ell \in (0, \frac{N}{2})$.

We now proceed by contradiction. Suppose the contrary. By continuity of the second eigenvalue, we should have for all $\epsilon > 0$

- either $\lambda_2(H(\epsilon, N))$ belongs to $\cup_{0<\ell<\frac{N}{2}} \sigma(H^{(\ell)}(\epsilon, N)))$

$$\text{and not to } \sigma(H^{(0)}(\epsilon, N)) \cup \sigma(H^{(N/2)}(\epsilon, N))\,,$$

- or $\lambda_2(H(\epsilon, N)) \in \sigma(H^{(0)}(\epsilon, N)) \cup \sigma(H^{(N/2)}(\epsilon, N))$

$$\text{and not to } \cup_{0<\ell<\frac{N}{2}} \sigma(H^{(\ell)}(\epsilon, N)))\,.$$

But, as we shall see later, the analysis for $\epsilon > 0$ small enough shows that we should be in the first case and the analysis for ϵ close to $\frac{\pi}{N}$ that we should be in the second case. Hence a contradiction.

The analysis for $\epsilon > 0$ small is by perturbation a consequence of the continuity and the analysis of $\epsilon = 0$. Here we see from (2) that $\lambda_2(D_{R_1,R_2})$ is simple and belongs to $\sigma(H^{(0)}(0,N))$.

If we only have (3), we observe that the two first eigenvalues belong to $\sigma(H^{(0)}(0,N))$ and the argument is unchanged.

The analysis for ϵ close to $\frac{\pi}{N}$ is by perturbation a consequence of the continuity and the analysis of $\epsilon = \frac{\pi}{N}$. Here we see from (14) that $\lambda_2(B_{R_2})$ has multiplicity two corresponding to $\sigma(H^{(1)}(\frac{\pi}{N},N))$.

So we have proven that for this value of ϵ the multiplicity is at least 3, hence equals 3 by the extension of Cheng's statement [Ch1976].

Theoretical asymptotics in domains with cracks

To simplify, we look at the case $N = 2$ and consider $0 < R_1 < R_2$. Motivated by the previous question, we analyze the different spectral problems according to the symmetries. This leads us to consider on the quarter of a disk ($0 < \theta < \frac{\pi}{2}$) four different models. On the exterior circle and on the cracks, we always assume the Dirichlet condition and then, according to the boundary conditions retained for $\theta = 0$ and $\theta = \pi/2$, we consider four test cases :

- Case NND (homogeneous Neumann boundary conditions for $\theta = 0$ and $\theta = \pi/2$).
- Case DDD (homogeneous Dirichlet boundary conditions for $\theta = 0$ and $\theta = \pi/2$).
- Case NDD (homogeneous Neumann boundary conditions for $\theta = 0$ and homogeneous Dirichlet boundary conditions for $\theta = \pi/2$).
- Case DND (homogeneous Dirichlet boundary conditions for $\theta = 0$ and homogeneous Neumann boundary conditions for $\theta = \pi/2$).

This is immediately related to the problem on the cracked disk by using the symmetries with respect to the two axes. The symmetry properties lead either to Dirichlet or Neumann.

The case DND

We use the notation

$$\begin{array}{ll} B^+_{R_2} & := B_{R_2} \cap \{x_2 > 0\}\,; \\ x_+ & := (0, R_1)\,; \\ \delta & := \frac{\pi}{2} - \epsilon\,; \\ K^+_\delta & = \{r = R_1, \theta \in [\frac{\pi}{2} - \delta, \frac{\pi}{2} + \delta] \end{array}$$

and look at the limit as $\delta \to 0$. By the symmetry arguments we have

$$\lambda_1^{DND}(\widehat{\mathfrak{D}}(2,\epsilon)) = \lambda_1(B^+_{R_2} \setminus K^+_\delta).$$

The family of compact sets $(K^+_\delta)_{\delta > 0}$ concentrates to the set $\{x_+\}$.

The reference Abatangelo-Felli-Hillairet-Léna [1] provides two-term asymptotic expansions in this situation.

Abatangelo-Felli-Hillairet-Léna [1] result

It gives

$$\lambda_1(B^+_{R_2}\backslash K^+_\delta) = \lambda_1(B^+_{R_2}) + u(x_+)^2 \frac{2\pi}{\left|\log(\mathrm{diam}(K^+_\delta)\right|} + o\left(\frac{1}{\left|\log(\mathrm{diam}(K^+_\delta)\right|}\right),$$

where $\mathrm{diam}(K^+_\delta)$ is the diameter of K^+_δ and u an eigenfunction associated with $\lambda_1(B^+_{R_2})$, normalized in $L^2(B^+_{R_2})$.
Using $\mathrm{diam}(K^+_\delta) = 2R_1 \sin(\delta)$ and the normalized eigenfunction corresponding to Dirichlet in $B^+_{R_2}$ we find as $\epsilon \to 0$

$$\begin{aligned} &\lambda_1^{DND}(\widehat{\mathfrak{D}}(2,\epsilon)) \\ &\quad = j_{1,1}^2 + + \tfrac{8}{R_2^2}\left(\tfrac{J_1(j_{1,1}R_1/R_2)}{J_1'(j_{1,1})}\right)^2 \tfrac{1}{|\log(\pi/2-\epsilon)|} \\ &\qquad + o\left(\tfrac{1}{|\log(\pi/2-\epsilon)|}\right), \end{aligned} \tag{15}$$

where $j_{\ell,k}$ is the k-th zero of the Bessel function J_ℓ corresponding to the integer $\ell \in \mathbb{N}$.

The case of the quarter of a disk (DND) with D-cracks

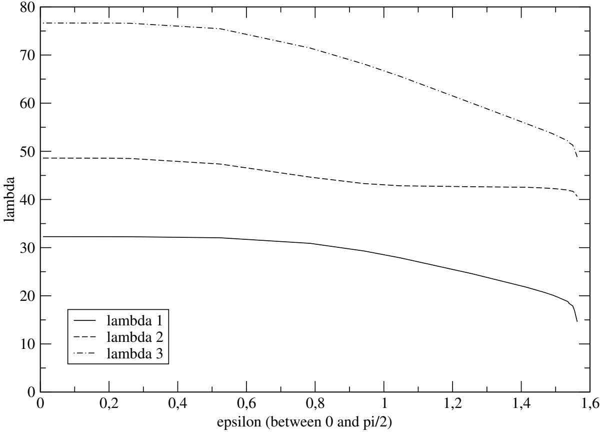

Figure: Case Dirichlet-Neumann : three first eigenvalues

THANK YOU FOR YOUR ATTENTION.

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