Adaptive Spectral Decomposition for Inverse Medium Problems

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Outline

1. Motivation
2. Adaptive Spectral Decomposition (ASD)
3. Approximation Theory
4. Numerical Examples
5. Adaptive Spectral Inversion (ASI)
6. Numerical Examples
7. Concluding Remarks

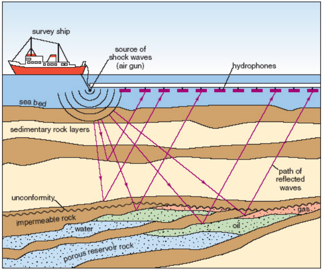
Motivation

Forward problem

Given a source f_ℓ, a medium u; y_ℓ satisfies the wave equation in time domain:

$$\begin{cases} \dfrac{\partial^2}{\partial t^2} y_\ell - \nabla \cdot (u(x) \nabla y_\ell) = f_\ell & \text{in } \Omega \times I, \\ IC(y_\ell(\cdot, T_1)) = 0 & \text{in } \Omega \\ BC(y_\ell) = 0 & \text{on } \partial\Omega \times I. \end{cases}$$

In a bounded spatial domain $\Omega \subset \mathbb{R}^d$ and time domain $I = (T_1, T_2) \subset \mathbb{R}$.



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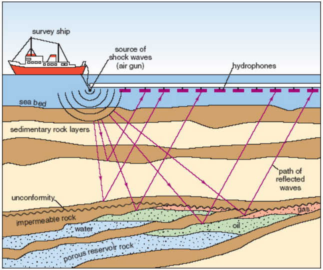
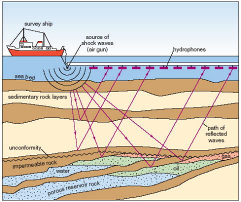
Inverse problem

Suppose the medium u inside Ω is illuminated by sources f_ℓ, with $\ell = 1, \dots, N_s$, and the responses y_ℓ^{obs} are recorded on $\Gamma \times I$, $\Gamma \subset \partial\Omega$.

Goal: find u





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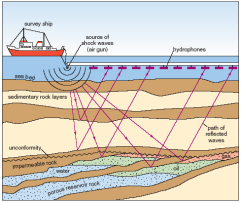
Formulation as an optimization problem:

$$u_* \in \arg\min_{v} \mathcal{J}(v),$$

where $\mathcal{J}$ is the (reduced) misfit

$$\mathcal{J}(v) = \frac{1}{2} \sum_{\ell=1}^{N_s} \int_{T_1}^{T_2} \left\| y_\ell(v) - y_\ell^{\mathrm{obs}} \right\|_{L^2(\Gamma)}^2 \, dt.$$



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Regularization:

Add a regularization term:

$$u_* \in \arg\min_{v} \left(\mathcal{J}(v) + \mathcal{R}(v) \right)$$

Finite dimensional search space:

$$u_* \in \arg\min_{v \in \varphi_0 + \Phi_K} \mathcal{J}(v)$$

Finite dimensional search space:

How to choose the space Φ_K, $(\dim \Phi_K = K)$?

- Discretization by, e.g., finite elements: $\Phi_K =$ space of discrete media
- Prior knowledge (regularization by parametrization)
- Adaptively

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  1. Choose initial search space $\varphi_0^{(1)} + \Phi^{(1)}$, $(\dim \Phi^{(1)} = K_1)$
  2. For $n \geq 1$
     - Solve
       $$u^{(n)} \in \underset{v \in \varphi_0^{(n)} + \Phi^{(n)}}{\arg\min} \mathcal{J}(v)$$
       e.g. with Newton-type method: BFGS, Gauss-Newton
     - choose new search space $\varphi_0^{(n+1)} + \Phi^{(n+1)}$
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Idea: Use current medium $u^{(n)}$ to construct $\varphi_0^{(n+1)} + \Phi^{(n+1)}$.

Adaptive Spectral Decomposition (ASD)

[De Buhan, Osses, 2010], [De Buhan, Kray, 2013], [G., Kray, Nahum, 2017], [Baffet, G., Tang, 2020], [Baffet, Gleichmann, G., 2021, *preprint*]:

Given an approximation $u^{(n-1)}$ of u_*, seek $u^{(n)}$ in

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where

$$L_\varepsilon[u^{(n-1)}]\varphi_0^{(n)} = 0 \quad \text{in } \Omega, \qquad \varphi_0^{(n)} = u^{(n-1)} \quad \text{on } \partial\Omega,$$

and for $k = 1, \ldots, K_n$

$$L_\varepsilon[u^{(n-1)}]\varphi_k^{(n)} = \lambda_k^{(n)}\varphi_k^{(n)} \quad \text{in } \Omega, \qquad \varphi_k^{(n)} = 0 \quad \text{on } \partial\Omega,$$

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with an elliptic operator

$$L_\varepsilon[w]v = -\nabla \cdot (\mu[w]\nabla v),$$
$$\mu[w](x) = \frac{1}{\sqrt{|\nabla w(x)|^2 + \varepsilon^2}},$$

$\varepsilon > 0$ to avoid dividing by 0. Typically we set $\varepsilon = 10^{-8}$.

Note that $\varphi_0^{(n)}$ and $\Phi^{(n)}$ depend on $u^{(n-1)}$.

Admissible approximation

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If ∇w is not well defined, e.g. w is piecewise constant, we need an admissible approximation $w_\delta \approx w$

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$$\lim_{\delta \to 0} \|w - w_\delta\|_{L^2(\Omega)} = 0, \quad \nabla w_\delta \in L^\infty(\Omega), \quad \mathrm{supp}(\nabla w_\delta) \subset \mathcal{M}_\delta,$$

where $\mathcal{M}_\delta$ is a δ neighborhood around the discontinuities of w, and there exists $C > 0$ such that for every $\delta > 0$: $\delta \|\nabla w_\delta\|_{L^\infty(\Omega)} \leq C$.

These properties hold, for instance, for

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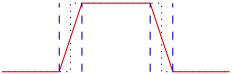
These properties hold, for instance, for

- the H^1-conforming $\mathcal{P}^r$, $r > 0$, FE interpolant on families of quasi-uniform meshes with mesh-size $h = \delta$

w piecewise constant

w_δ as $\mathcal{P}^1$ interpolant



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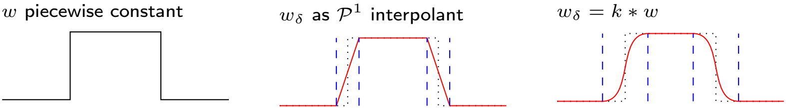
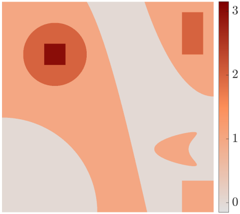
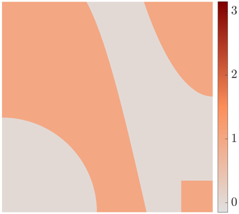
- the convolution of w with a smoothing kernel (mollifier)



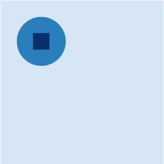
Example (numerical)



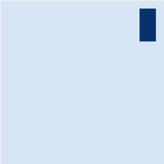
(a) Medium u



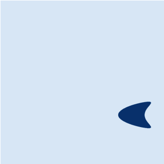
(c) φ_0



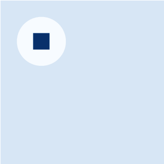
(a) φ_1



(b) φ_2



(c) φ_3



(d) φ_4

$$L_\varepsilon[u_\delta]v = -\nabla \cdot (\mu(u_\delta)\nabla v), \qquad \mu[u_\delta](x) = \frac{1}{\sqrt{|\nabla u_\delta(x)|^2 + \varepsilon^2}}, \quad \varepsilon > 0$$

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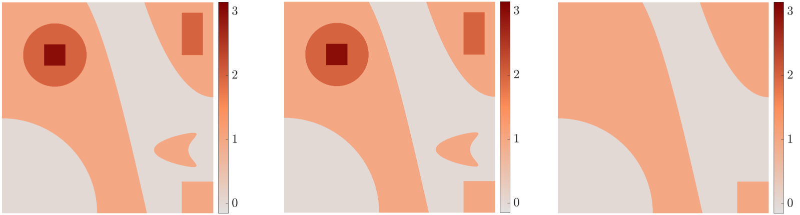
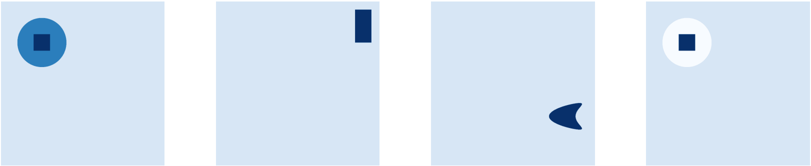


(a) Medium u (b) Best L^2 approximation (c) φ_0



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Link to Total Variation

- For smooth functions, the total variation functional is given by

$$\mathrm{TV}(v) = \|\nabla v\|_{L^1(\Omega)} = \int_\Omega |\nabla v|$$

and its Fréchet derivative by

$$D\,\mathrm{TV}(v) = -\nabla \cdot \left(\frac{\nabla v}{|\nabla v|} \right).$$

Commonly used in image processing for edge-preserving noise removal [Rudin, Osher, Fatemi, 1992], etc.

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- For the "smoothed" TV functional

$$\mathrm{TV}_\varepsilon(v) = \int_\Omega \sqrt{|\nabla v|^2 + \varepsilon^2}$$

we have

$$D\,\mathrm{TV}_\varepsilon(v) = -\nabla \cdot \left(\frac{\nabla v}{\sqrt{|\nabla v|^2 + \varepsilon^2}} \right) = L_\varepsilon[v]v,$$

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G., Kray, Nahum, Inverse Problems 33, 2017

Approximation theory

Goal: Given u piecewise constant; understand the behavior of the first eigenfunctions of $L_\varepsilon[u_\delta]$ for $u_\delta \approx u$ and the accuracy of the best L^2 approximation of u in $\varphi_0 + \Phi_K$.

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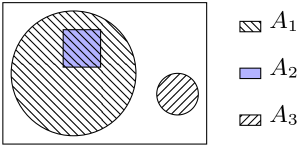
Definitions:
For simplicity $u = 0$ near $\partial\Omega$.
Let $\Omega \subset \mathbb{R}^d$ be open, bounded and with Lipschitz boundary,

$$u(x) = \sum_{k=1}^{K} \alpha_k \chi_{A_k}(x), \quad \alpha_k \neq 0,$$

where χ_{A_k} is the characteristic function of a Lipschitz domain $A_k \subset\subset \Omega$ with connected and mutually disjoint boundaries.



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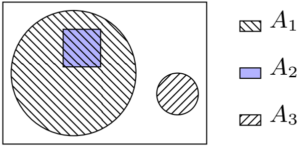
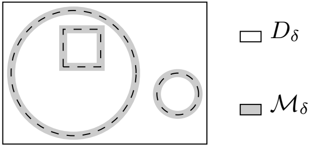
For each $\delta > 0$ let

$$\mathcal{M}_\delta = \bigcup_{k=1}^{K} \{x \in \Omega \,|\, \operatorname{dist}(x, \partial A_k) < \delta\}, \qquad D_\delta = \Omega \setminus \overline{\mathcal{M}_\delta}.$$





Consider a closed (finite or infinite) subspace $\mathcal{V}_0^\delta \subset H_0^1(\Omega)$.

Let $\lambda \in \mathbb{R}$ and $\varphi \in \mathcal{V}_0^\delta$ be an eigenvalue and eigenfunction of $L_\varepsilon[u_\delta]$ in $\mathcal{V}_0^\delta$, i.e.

$$B[\varphi, w] = \lambda \langle \varphi, w \rangle, \qquad \forall w \in \mathcal{V}_0^\delta$$

where

$$B[v, w] = \langle \mu[u_\delta] \nabla v, \nabla w \rangle.$$

Since $L_\varepsilon[u_\delta]$ is elliptic

- $(\lambda_k)_k$ are the nondecreasing eigenvalues of $L_\varepsilon[u_\delta]$ in $\mathcal{V}_0^\delta$ with each eigenvalue repeated according to its multiplicity,
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Theorem [Baffet, G., Tang, 2020]

Let $u = \sum_{k=1}^K \alpha_k \chi_{A_k}$ and u_δ be an admissible approximation of u and let φ_k be the first K eigenfunctions of $L_\varepsilon[u_\delta]$ for $\varepsilon, \delta > 0$.

Then there exists a constant $C > 0$ such that for ε, δ sufficiently small

$$\|\nabla \varphi_k\|_{L^2(D_\delta)} \le C\sqrt{\varepsilon}.$$

Essentially, φ_k, $k = 1, \ldots, K$, of $L_\varepsilon[u_\delta]$ are "almost" constant.
$\implies \Phi_K = \mathrm{span}(\varphi_k)_{k=1}^K$ should approximate u well.

Adaptive spectral decomposition

Consider $u = \sum_{k=1}^{K} \alpha_k \chi_{A_k}$

1. Approximate u by u_δ
2. Compute the first K eigenfunctions φ_k of $L_\varepsilon[u_\delta]$
3. Project u into $\Phi_K = \mathrm{span}(\varphi_k)_{k=1}^{K}$ to obtain $\Pi_K^\varepsilon[u_\delta]u \in \mathcal{V}_0^\delta$ via the standard orthogonal projection:

$$\Pi_K^\varepsilon[u_\delta] : L^2(\Omega) \to \Phi_K, \qquad \langle v - \Pi_K^\varepsilon[u_\delta]v, \varphi\rangle = 0 \quad \forall \varphi \in \Phi_K.$$

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Theorem [Baffet, Gleichmann, G., 2021, *preprint*]

Let $u = \sum_{k=1}^{K} \alpha_k \chi_{A_k}$ and u_δ be an admissible approximation of u, $(\varphi_k)_k$ the first K eigenfunctions of $L_\varepsilon[u_\delta]$ for $\varepsilon, \delta > 0$.
Let $\Pi_K^\varepsilon[u_\delta]$ be the L^2 orthogonal projection on Φ_K.

Then, for every $v \in \operatorname{span}(\chi_{A_k})_{k=1}^{K}$, there exists a constant $C = C(v) > 0$ such that for ε, δ sufficiently small

$$\|v - \Pi_K^\varepsilon[u_\delta]v\|_{L^2(\Omega)} \leq C\sqrt{\varepsilon + \delta}.$$

In particular the above is true for $v = u$.

In a nutshell:

A medium $u(x)$ with K piecewise constant inclusions can be approximated arbitrarily well as a linear combination of the first K eigenfunctions of $L_\varepsilon[u_\delta]$, with $u_\delta \approx u$.

In practice, the eigenfunctions are computed numerically, e.g. with finite elements and Matlab.

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Numerical Examples

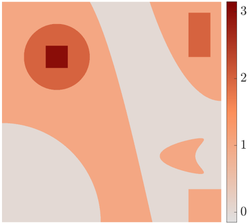
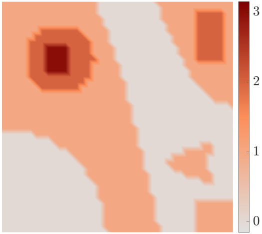
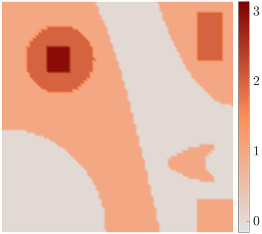


Figure: Medium u

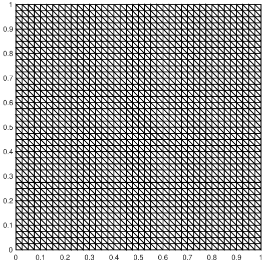
Numerical Examples



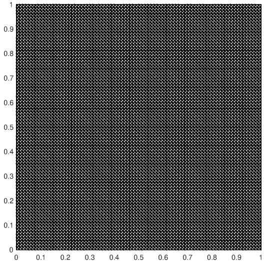
(a) u_δ with $\delta = 0.05 \cdot 2^{-1}$



(b) u_δ with $\delta = 0.05 \cdot 2^{-2}$



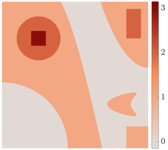
(c) FE mesh with $h = \delta = 0.05 \cdot 2^{-1}$



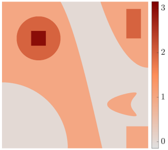
(d) FE mesh with $h = \delta = 0.05 \cdot 2^{-2}$

Numerical Examples

L^2 **estimate:** verification of $\|u - \Pi_K^\varepsilon[u_\delta]u\|_{L^2(\Omega)} = \mathcal{O}(\sqrt{\varepsilon + \delta})$

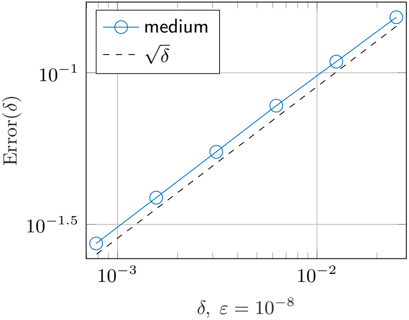


(a) Medium u

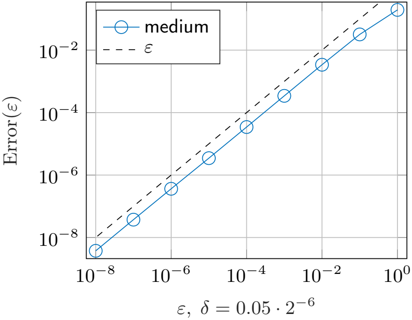


(b) Best L^2 approximation

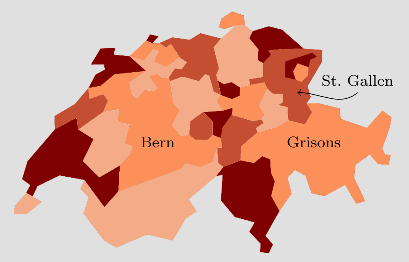
$\text{Error}(\delta) = \|u - \Pi_K^\varepsilon[u_\delta]u\|_{L^2(\Omega)}$:



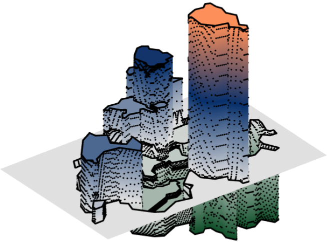
$\text{Error}(\varepsilon) = \|u_\delta - \Pi_K^\varepsilon[u_\delta]u_\delta\|_{L^2(\Omega)}$:



Polygonal map of Switzerland

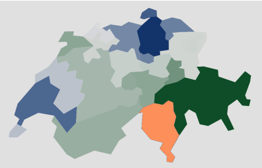


(a) Polygonal Switzerland u_{CH}

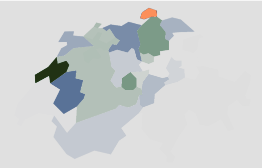


(b) 3D-view of φ_5

(c) φ_2



(d) φ_5



(e) φ_{15}

Polygonal map of Switzerland

Given the map of Switzerland in a polygonal form such that

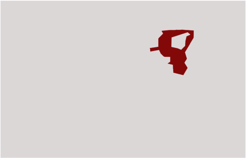
$$u_{\mathrm{CH}} = \sum_{k=1}^{26} \alpha_k \chi_{A_k}$$

where each A_k (may) represents a single Canton. Compute the first 26 eigenfunctions of $L_\varepsilon[u_{\mathrm{CH}}]$.

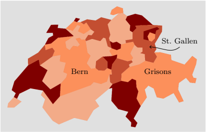
$\Longrightarrow$ we are able to project each Canton into $\mathrm{span}(\varphi_k)_{k=1}^{26}$:

(a) Canton of Bern

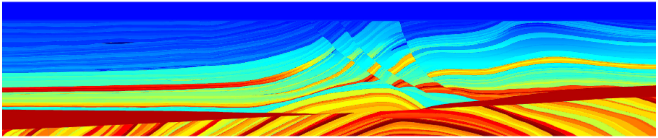
(b) Canton of Grisons



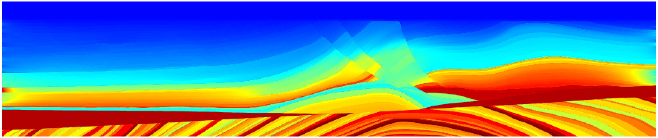
(c) Canton of St. Gallen



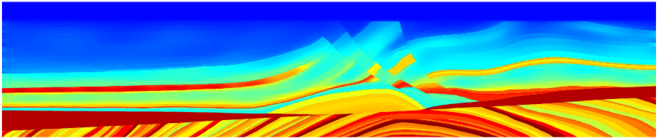
Marmousi model [Martin, Wiley, Marfurt, 2006]



(a) The Marmousi model



(b) φ_0 with $e_{\text{rel}} \approx 12.89\,\%$



(c) AS projection with $K = 100$ and $e_{\text{rel}} \approx 3.8\,\%$

Adaptive Spectral Inversion

Goal: find $u_* \in \arg\min_v \mathcal{J}(v)$

For simplicity $u = 0$ near $\partial\Omega$ $\quad$ ($\implies \varphi_0^{(n)} = 0 \ \forall n$)

1. Choose initial search space $\Psi^{(1)} = \mathrm{span}\{\psi_1^{(1)}, \dots, \psi_{K_1}^{(1)}\}$
2. For $n \geq 1$
   - *Solve:*
   $$u^{(n)} \in \arg\min_{v \in \Psi^{(n)}} \mathcal{J}(v)$$
   - *Compute:* the first eigenfunctions
   $$\varphi_1^{(n+1)}, \dots, \varphi_{K_n}^{(n+1)} \qquad \text{of } L_\varepsilon[u^{(n)}]$$
   - *Merge:* compute an orthonormal basis $\Phi^{(n+1)}$ of
   $$\mathrm{span}\left\{\varphi_1^{(n+1)}, \dots, \varphi_{K_n}^{(n+1)}, \psi_1^{(n)}, \dots, \psi_{K_n}^{(n)}\right\}$$
   - *Reduce:* new basis/search space $\Psi^{(n+1)} \subset \Phi^{(n+1)}$ such that
   $$\operatorname*{proj}_{\Psi^{(n+1)}} u^{(n)} \approx u^{(n)}$$
3. End

Numerical Examples

Adaptive spectral inversion for the wave equation in time domain with absorbing boundary conditions.

Parameter settings:

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- $N_s = 32$ evenly distributed sources near the boundary
- SAA (sample average approximation) approach with only one single "super shot" [Haber, Chung, Herrman, 2012] $\implies$ computational cost reduced by nearly $1/32$

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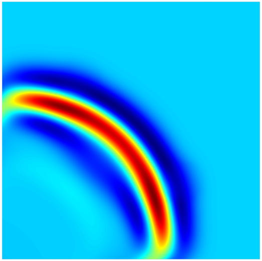
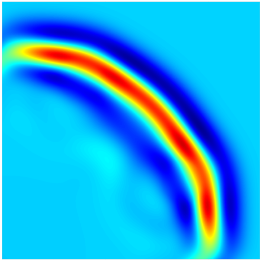
- About 50 BFGS iterations per optimization step
- Stop when the discrepancy principle is satisfied: given noisy data y^{obs} with $\|y^{\mathrm{true}} - y^{\mathrm{obs}}\| \leq \eta$ then stop in iteration n_* when

$$\|y(u_{n_*}) - y^{\mathrm{obs}}\| \leq \tau\eta, \qquad \tau > 1.$$

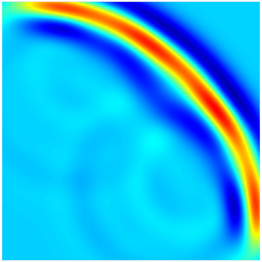
Solution to the forward problem



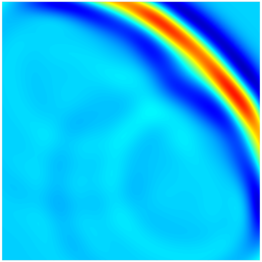
(a) solution at time $t = 0.7$



(b) solution at time $t = 0.9$

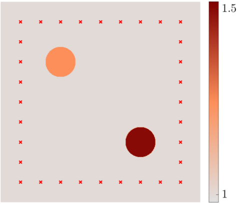


(c) solution at time $t = 1.1$



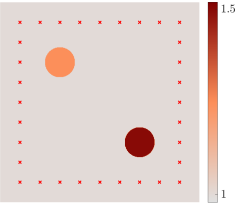
(d) solution at time $t = 1.2$

Two discs

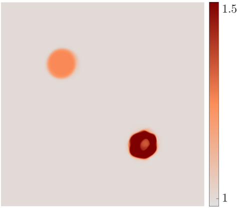


(a) u_{true}

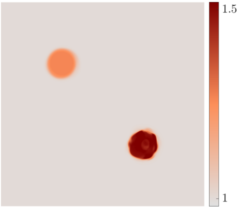
Two discs



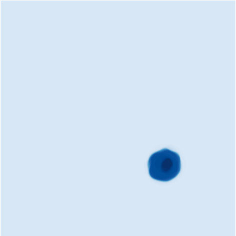
(a) u_{true}



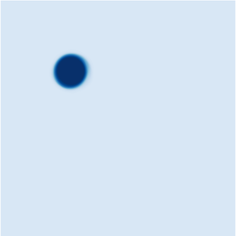
(b) u_*, $e_{\text{rel}} \approx 2.07\,\%$



(c) $\Pi^{\varepsilon}_{K_*}[u_*]u_{\text{true}}$



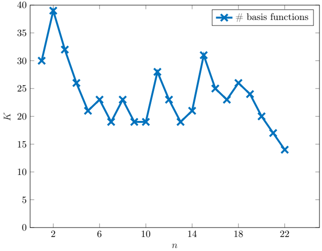
(d) $\psi_1^{(*)}$



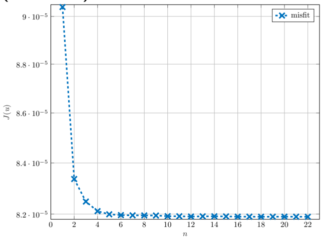
(e) $\psi_2^{(*)}$

Two discs

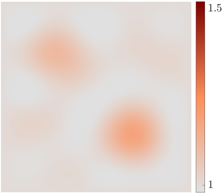
Number of basis functions:



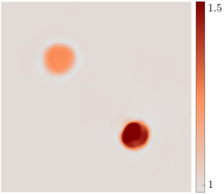
(reduced) misfit:



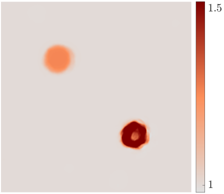
Snapshots of the AS iterates:



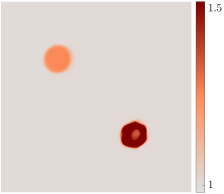
(a) $u^{(1)}$



(b) $u^{(5)}$

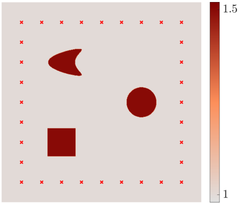


(c) $u^{(15)}$



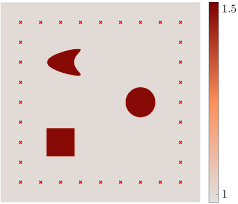
(d) $u_* = u^{(22)}$

Three inclusions



(a) u_{true}

Three inclusions



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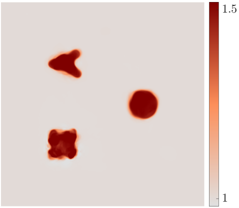
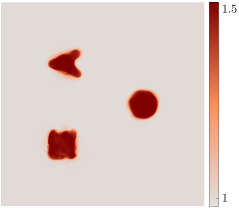
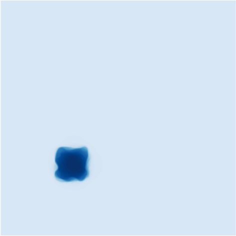


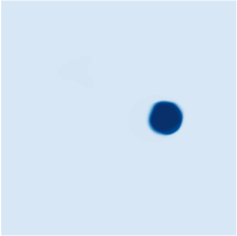
(b) u_*, $e_{\text{rel}} \approx 2.96\,\%$



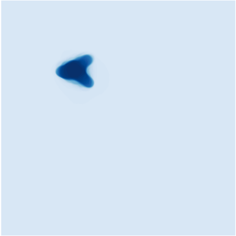
(c) $\Pi^{\varepsilon}_{K_*}[u_*]u_{\text{true}}$



(d) $\psi_1^{(*)}$



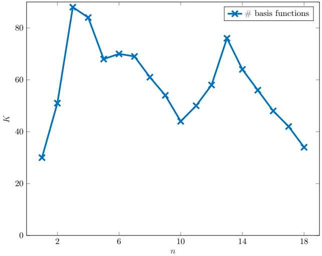
(e) $\psi_2^{(*)}$



(f) $\psi_3^{(*)}$

Three inclusions

Number of basis functions:



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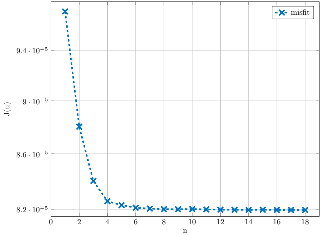

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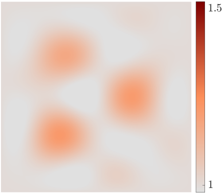
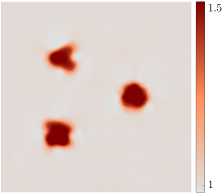
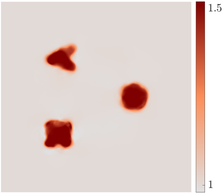


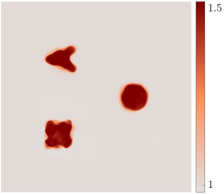
(a) $u^{(1)}$



(b) $u^{(5)}$



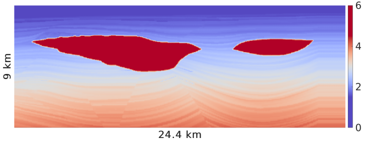
(c) $u^{(10)}$



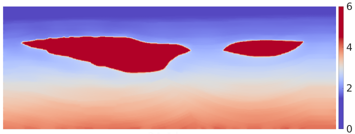
(d) $u_* = u^{(18)}$

Subsalt FWI: Pluto 1.5 model[1] (Helmholtz Equation / frequency stepping):

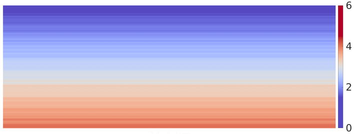
Medium



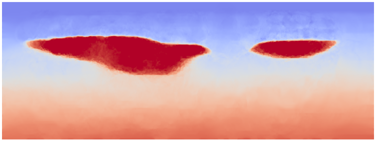
Best L^2 approx. in $\varphi_0 + \Phi_K$, with $K = 100$



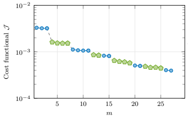
Initial guess (borehole data at $x = x_E$)



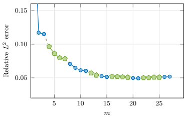
ASI solution (surface data, 20% noise)



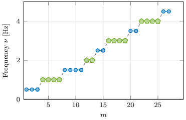
Misfit



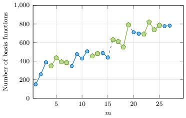
L^2 **error**



Frequency



Number of basis functions



u_h (P^1-FE), y_h (P^3-FE), 351'360 elements,

(Baffet, G., Tang, Inv. Probl. 37, 2021).

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[1] `http://epos-eu.cz/ssc/software/sw3d/data/plu/plu.htm`

Concluding remarks

- Adaptive spectral inversion/decomposition
  - efficient way to represent piecewise constant media (K inclusions $\rightarrow$ K eigenfunctions)
  - much smaller number of control variables K
  - efficient alternative to standard Tikhonov regularization
  - robust to missing data or added noise
- Extra cost
  - φ_0: solve elliptic (coercive) PDE at each basis update
  - φ_k: compute the first K eigenfunctions (Lanczos) of elliptic (coercive) operator
- Extensions
  - applies to multi-parameter inversion, too
  - Connection to (nonlinear) spectral decomposition in imaging (Gilboa et al.)
  - Adaptive spectral representation independent of model problem (Helmholtz, wave equation), i.e. useful for other inverse problems, too

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