

Multi-Trace FEM-BEM formulations for acoustic scattering by composite objects

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The logo for Inria, featuring the word "Inria" in a stylized, red, cursive script.

Outline

- 1 Introduction
- 2 Recap of boundary integral operators and Costabel coupling
- 3 Single-Trace FEM-BEM formulation
- 4 Multi-Trace FEM-BEM formulation
- 5 Combined field formulations

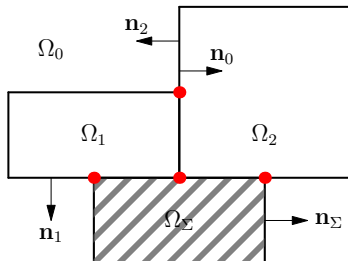
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Multi-domain acoustic scattering

The transmission problem

Find $U \in H_{\text{loc}}^1(\mathbb{R}^d)$ s.t.

$$\left\{ \begin{array}{l} -\Delta U - \kappa_{\Sigma}^2(\mathbf{x})U = f \text{ in } \Omega_{\Sigma} \\ -\Delta U - \kappa_j^2 U = 0 \text{ in } \Omega_j, j = 0, \dots, n \\ U - U_{\text{inc}} \text{ outgoing in } \Omega_0 \\ \gamma_{\text{D}}^j U - \gamma_{\text{D}}^k U = 0 \\ \gamma_{\text{N}}^j U + \gamma_{\text{N}}^k U = 0 \quad \text{on } \partial\Omega_j \cap \partial\Omega_k, j \neq k \\ \gamma_{\text{D}}^j U - \gamma_{\text{D}}^{\Sigma} U = 0 \\ \gamma_{\text{N}}^j U + \gamma_{\text{N}}^{\Sigma} U = 0 \quad \text{on } \partial\Omega_j \cap \partial\Omega_{\Sigma} \end{array} \right.$$



Trace operators (by density)

$$\gamma_{\text{D}}^j \varphi := \varphi|_{\partial\Omega_j}$$

$$\gamma_{\text{N}}^j \varphi := \mathbf{n}_j \cdot \nabla \varphi|_{\partial\Omega_j}$$

⚠ Junction points ⚠

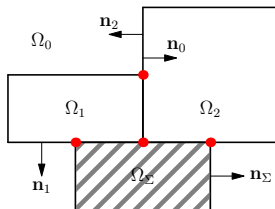
Goal: stable formulation with **FEM** in Ω_{Σ} + **BEM** in $\Omega_j, j = 0, \dots, n$

FEM vs BEM: benefit from both!

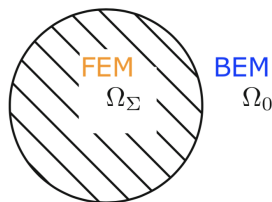
Finite Element Method (**FEM**): discretization of a partial differential equation

Boundary Element Method (**BEM**): discretization of a **Boundary Integral Equation (BIE)**: reformulation on the boundary of the object

- surface mesh \Rightarrow **reduction** in the number of **unknowns**,
- no need of artificial boundary conditions for an **unbounded domain**,
- non local integral kernels \Rightarrow **dense matrices**,
- possible only if **piecewise constant** propagation medium.

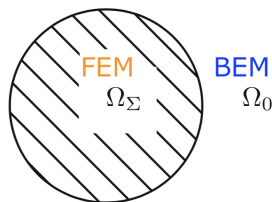


FEM-BEM coupling strategies (2 domains)



- **Johnson-Nédélec coupling:** direct BIE in Ω_0
- **Costabel coupling:** direct BIE in Ω_0 , symmetric!
- **Bielak-MacCamy coupling:** indirect BIE in Ω_0

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Boundary integral formulations for multi-domain setting

- **Single-Trace Formulation (STF)** (also called Rumsey or PMCHWT):
only one pair of traces at each point of each interface

[Rumsey, 1954], [Poggio & Miller, 1973], [Chang & Harrington, 1977], [VonPetersdorff, 1989] ...

- **Boundary Element Tearing and Interconnecting method (BETI)**:
boundary integral equation version of FETI

[Langer & Steinbach, 2003], [Langer & al, 2007], [Steinbach & Of, 2009] ...

- STF of **2nd kind**: multi-domain version of Müller formulation with non-singular integral kernels

[Claeys, 2011], [Greengard & Lee, 2012], [Claeys, Hiptmair & Spindler, 2017]

- **Multi-Trace Formulation (MTF)** (global, quasi-local, local):
pair of traces are *doubled* on each interface

[Hiptmair & Jerez-Hanckes, 2012], [Claeys & Hiptmair, 2013], [Claeys, 2015] ...

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With junction points

- mainly explored strategy is FETI-BETI
- here we choose to study STF (or MTF) for the BEM part:
proper functional spaces even with junction points

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Pair of traces and potentials

$$\mathbb{H}(\partial\Omega_j) = H^{1/2}(\partial\Omega_j) \times H^{-1/2}(\partial\Omega_j)$$

Interior and exterior **trace operators**:

$$\gamma^j V = \begin{pmatrix} \gamma_D^j V \\ \gamma_N^j V \end{pmatrix} \in \mathbb{H}(\partial\Omega_j) \quad \gamma_c^j V = \begin{pmatrix} \gamma_{D,c}^j V \\ \gamma_{N,c}^j V \end{pmatrix} \in \mathbb{H}(\partial\Omega_j)$$

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$\mathcal{G}_\kappa(\mathbf{x})$: Green kernel for the Helmholtz operator $-\Delta - \kappa^2$ (κ constant)

Potential operator (DL + SL potentials): for $\mathbf{v}_j = \begin{pmatrix} v \\ q \end{pmatrix} \in \mathbb{H}(\partial\Omega_j)$

$$G_\kappa^j(\mathbf{v}_j)(\mathbf{x}) = \int_{\partial\Omega_j} v(\mathbf{y}) \mathbf{n}_j(\mathbf{y}) \cdot (\nabla \mathcal{G}_\kappa)(\mathbf{x} - \mathbf{y}) d\sigma(\mathbf{y}) + \int_{\partial\Omega_j} q(\mathbf{y}) \mathcal{G}_\kappa(\mathbf{x} - \mathbf{y}) d\sigma(\mathbf{y})$$

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Representation formula

If $U \in H_{\text{loc}}^1(\bar{\Omega}_j)$ with $-\Delta U - \kappa^2 U = 0$ in Ω_j (outgoing if $j = 0$) then

$$U(\mathbf{x}) \mathbf{1}_{\Omega_j}(\mathbf{x}) = G_\kappa^j(\gamma^j U)(\mathbf{x})$$

Calderón identities

Take interior traces of repr. formula $G_{\kappa}^j(\gamma^j U)(\mathbf{x}) = U(\mathbf{x})$:

Calderón identities

$$\gamma^j \circ G_{\kappa}^j(\gamma^j U) = \gamma^j U$$

characterizes traces of solutions to homogeneous Helmholtz eq!

$\gamma^j \circ G_{\kappa}^j$: Calderón projector

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Boundary integral operators:

$$A_{\kappa}^j = \{\gamma^j\} \circ G_{\kappa}^j = \frac{1}{2}(\gamma^j + \gamma_c^j) \circ G_{\kappa}^j = \begin{pmatrix} K_j & V_j \\ W_j & K_j' \end{pmatrix}$$

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By jump relations ...

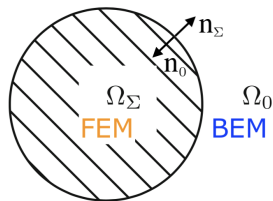
$$\gamma^j \circ G_{\kappa}^j = A_{\kappa}^j + \text{Id}/2$$

so the characterization becomes

$$(A_{\kappa}^j - \text{Id}/2)(\gamma^j U) = 0$$

Classical Costabel coupling: 2 domains

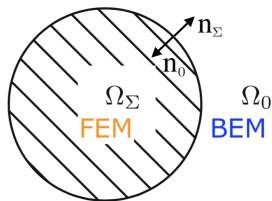
$$n = 0 \quad \mathbb{R}^d = \bar{\Omega}_0 \cup \bar{\Omega}_\Sigma \quad \Gamma = \partial\Omega_0 = \partial\Omega_\Sigma$$



$$\left\{ \begin{array}{l} -\Delta U - \kappa_\Sigma^2(\mathbf{x})U = f \text{ in } \Omega_\Sigma \\ -\Delta U - \kappa_0^2 U = 0 \text{ in } \Omega_0 \\ U - U_{\text{inc}} \text{ outgoing in } \Omega_0 \\ \gamma_D^0 U - \gamma_D^\Sigma U = 0 \\ \gamma_N^0 U + \gamma_N^\Sigma U = 0 \end{array} \right. \text{ on } \Gamma$$

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In Ω_Σ : find $U \in H^1(\Omega_\Sigma)$ s.t.

$$\langle v, q \rangle_\Gamma = \int_\Gamma vq \, d\sigma$$

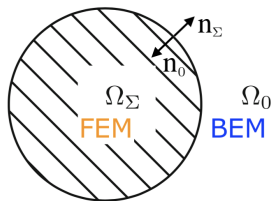
$$\int_{\Omega_\Sigma} (\nabla U \cdot \nabla V - \kappa_\Sigma^2(\mathbf{x})UV) \, d\mathbf{x} - \langle \gamma_D^\Sigma V, \gamma_N^\Sigma U \rangle_\Gamma = \int_{\Omega_\Sigma} fV \, d\mathbf{x} \quad \forall V \in H^1(\Omega_\Sigma)$$

using $\gamma_N^0 U = -\gamma_N^\Sigma U$:

$$a_\Sigma(U, V) + \langle \gamma_D^\Sigma V, \gamma_N^0 U \rangle_\Gamma = F_\Sigma(V) \quad \forall V \in H^1(\Omega_\Sigma)$$

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Classical Costabel coupling: 2 domains

In Ω_0 : *both* Calderón identities

$$\gamma^0(U - U_{\text{inc}}) = \gamma^0 G_{\kappa_0}^0(\gamma^0(U - U_{\text{inc}}))$$

$$\iff \begin{cases} \gamma_D^0 U - \gamma_D^0 G_{\kappa_0}^0 \begin{pmatrix} \gamma_D^0 U \\ \gamma_N^0 U \end{pmatrix} = \gamma_D^0 U_{\text{inc}} \\ \gamma_N^0 U - \gamma_N^0 G_{\kappa_0}^0 \begin{pmatrix} \gamma_D^0 U \\ \gamma_N^0 U \end{pmatrix} = \gamma_N^0 U_{\text{inc}} \end{cases}$$

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using $\gamma_D^0 U = \gamma_D^\Sigma U$, unknown $p := \gamma_N^0 U$.

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using $\gamma_D^0 U = \gamma_D^\Sigma U$, unknown $p := \gamma_N^0 U$.

Find $p \in H^{-\frac{1}{2}}(\Gamma)$ s.t.

$$\langle \gamma_D^\Sigma U, q \rangle_\Gamma - \left\langle \gamma_D^0 G_{\kappa_0}^0 \begin{pmatrix} \gamma_D^\Sigma U \\ p \end{pmatrix}, q \right\rangle_\Gamma = \langle \gamma_D^0 U_{\text{inc}}, q \rangle_\Gamma \quad \forall q \in H^{-\frac{1}{2}}(\Gamma)$$

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$$\gamma^0(U - U_{\text{inc}}) = \gamma^0 G_{\kappa_0}^0(\gamma^0(U - U_{\text{inc}}))$$

$$\Leftrightarrow \begin{cases} \gamma_D^\Sigma U - \gamma_D^0 G_{\kappa_0}^0 \begin{pmatrix} \gamma_D^\Sigma U \\ p \end{pmatrix} = \gamma_D^0 U_{\text{inc}} \\ \gamma_N^0 U - \gamma_N^0 G_{\kappa_0}^0 \begin{pmatrix} \gamma_D^\Sigma U \\ p \end{pmatrix} = \gamma_N^0 U_{\text{inc}} \end{cases} \quad \text{plug into FEM var. form.}$$

using $\gamma_D^0 U = \gamma_D^\Sigma U$, unknown $p := \gamma_N^0 U$.

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Classical Costabel coupling: 2 domains

After subtracting and simplifying...

defining the skew-symmetric duality pairing for $\begin{pmatrix} u \\ p \end{pmatrix}, \begin{pmatrix} v \\ q \end{pmatrix} \in \mathbb{H}(\Gamma)$

$$\left[\begin{pmatrix} u \\ p \end{pmatrix}, \begin{pmatrix} v \\ q \end{pmatrix} \right]_{\Gamma} := \langle u, q \rangle_{\Gamma} - \langle v, p \rangle_{\Gamma}$$

Costabel coupling

find $U \in H^1(\Omega_{\Sigma})$, $p \in H^{-1/2}(\Gamma)$ such that

$$\begin{aligned} a_{\Sigma}(U, V) + \left[A_{\kappa_0}^0 \begin{pmatrix} \gamma_D^{\Sigma} U \\ p \end{pmatrix}, \begin{pmatrix} -\gamma_D^{\Sigma} V \\ q \end{pmatrix} \right]_{\Gamma} - \frac{1}{2} \left[\begin{pmatrix} \gamma_D^{\Sigma} U \\ p \end{pmatrix}, \begin{pmatrix} \gamma_D^{\Sigma} V \\ q \end{pmatrix} \right]_{\Gamma} \\ = F_{\Sigma}(V) - \left[\gamma^0 U_{\text{inc}}, \begin{pmatrix} -\gamma_D^{\Sigma} V \\ q \end{pmatrix} \right]_{\Gamma} \quad \forall V \in H^1(\Omega_{\Sigma}), q \in H^{-1/2}(\Gamma) \end{aligned}$$

2×2 operator $A_{\kappa_0}^0$: explicit classical boundary integral operators

Classical Costabel coupling: 2 domains

Generalized Gårding inequality

\mathbf{a}_C : the bilinear form of Costabel coupling.

There exist a compact bilinear form \mathbf{k} , $\beta > 0$ s.t.

$$\operatorname{Re} \{ \mathbf{a}_C((V, q), (\overline{V}, \overline{q})) + \mathbf{k}((V, q), (\overline{V}, \overline{q})) \} \geq \beta (\|V\|_{H^1(\Omega_\Sigma)}^2 + \|q\|_{H^{-1/2}(\Gamma)}^2)$$

for all $V \in H^1(\Omega_\Sigma)$, $q \in H^{-\frac{1}{2}}(\Gamma)$.

\Rightarrow bijective if and only if injective!

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Spurious resonances

Let $U \in H^1(\Omega_\Sigma)$, $p \in H^{-\frac{1}{2}}(\Gamma)$ solve Costabel formulation with $f = 0$, $U_{\text{inc}} = 0$.

Then

- $U = 0$
- $p = 0$ IFF κ_0^2 is not an interior Dirichlet eigenvalue of $-\Delta$ on Ω_Σ , i.e.

$$\nexists W \in H^1(\Delta, \Omega_\Sigma) \setminus \{0\} \text{ such that } \begin{cases} -\Delta W = \kappa_0^2 W & \text{in } \Omega_\Sigma \\ W = 0 & \text{on } \partial\Omega_\Sigma. \end{cases}$$

See well-posed CFIE formulation in [Hiptmair & Meury, 2006]

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Trace spaces for multi-domain scattering

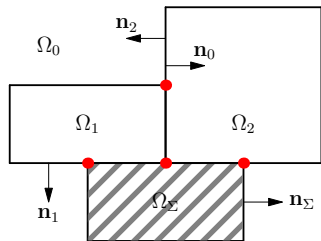
$$\mathbb{R}^d = \bigcup_{j=0}^n \bar{\Omega}_j \cup \bar{\Omega}_\Sigma$$

$$\Gamma = \bigcup_{j=0}^n \partial\Omega_j \quad \text{skeleton} \quad \Sigma = \partial\Omega_\Sigma$$

$$\mathbb{H}(\partial\Omega_j) = \mathbb{H}^{+1/2}(\partial\Omega_j) \times \mathbb{H}^{-1/2}(\partial\Omega_j)$$

$$\text{for } \mathbf{u}_j = (u, \rho), \mathbf{v}_j = (v, q) \in \mathbb{H}(\partial\Omega_j)$$

$$[\mathbf{u}_j, \mathbf{v}_j]_j = \langle u, q \rangle_j - \langle v, \rho \rangle_j$$



Trace spaces for multi-domain scattering

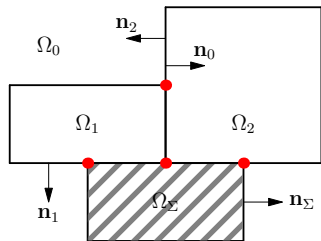
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Multi-trace space: $\mathbb{H}(\Gamma) = \mathbb{H}(\partial\Omega_0) \times \dots \times \mathbb{H}(\partial\Omega_n)$

Duality pairing:

$$\text{for } \mathbf{u} = (u_0, \dots, u_n), \mathbf{v} = (v_0, \dots, v_n) \in \mathbb{H}(\Gamma) \quad [\mathbf{u}, \mathbf{v}] = \sum_{j=0}^n [\mathbf{u}_j, \mathbf{v}_j]_j$$

Trace spaces for multi-domain scattering

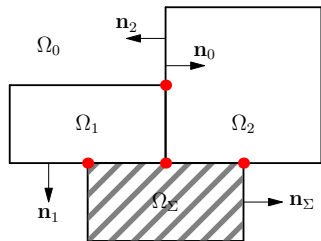
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Single-trace space $\mathbb{X}(\Gamma) \subset \mathbb{H}(\Gamma)$:

$\mathbb{X}(\Gamma)$ = tuples of traces that match the **transmission conditions**

Trace spaces for multi-domain scattering

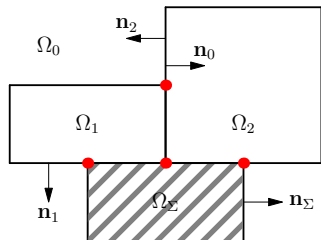
$$\mathbb{R}^d = \bigcup_{j=0}^n \bar{\Omega}_j \cup \bar{\Omega}_\Sigma$$

$$\Gamma = \bigcup_{j=0}^n \partial\Omega_j \quad \text{skeleton} \quad \Sigma = \partial\Omega_\Sigma$$

$$\mathbb{H}(\partial\Omega_j) = \mathbb{H}^{+1/2}(\partial\Omega_j) \times \mathbb{H}^{-1/2}(\partial\Omega_j)$$

$$\text{for } \mathbf{u}_j = (u, p), \mathbf{v}_j = (v, q) \in \mathbb{H}(\partial\Omega_j)$$

$$[\mathbf{u}_j, \mathbf{v}_j]_j = \langle u, q \rangle_j - \langle v, p \rangle_j$$



Multi-trace space: $\mathbb{H}(\Gamma) = \mathbb{H}(\partial\Omega_0) \times \dots \times \mathbb{H}(\partial\Omega_n)$

Duality pairing:

$$\text{for } \mathbf{u} = (\mathbf{u}_0, \dots, \mathbf{u}_n), \mathbf{v} = (\mathbf{v}_0, \dots, \mathbf{v}_n) \in \mathbb{H}(\Gamma) \quad [\mathbf{u}, \mathbf{v}] = \sum_{j=0}^n [\mathbf{u}_j, \mathbf{v}_j]_j$$

Single-trace space $\mathbb{X}(\Gamma) \subset \mathbb{H}(\Gamma)$:

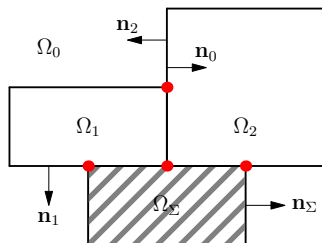
$\mathbb{X}(\Gamma)$ = tuples of traces that match the **transmission conditions**

$$\mathbb{X}(\Gamma) = \{ \mathbf{u} = (u_j, p_j)_{j=0}^n \mid (u_j)_{j=0}^n \in \mathbb{X}^{+1/2}(\Gamma), (p_j)_{j=0}^n \in \mathbb{X}^{-1/2}(\Gamma) \}$$

$$\mathbb{X}^{+1/2}(\Gamma) = \{ (V|_{\partial\Omega_j})_{j=0}^n \mid V \in \mathbb{H}^1(\mathbb{R}^d) \}$$

$$\mathbb{X}^{-1/2}(\Gamma) = \{ (\mathbf{n}_j \cdot \mathbf{q}|_{\partial\Omega_j})_{j=0}^n \mid \mathbf{q} \in \mathbb{H}^1(\text{div}, \mathbb{R}^d) \}$$

Trace spaces for multi-domain scattering



Every $\mathbf{x} \in \Sigma = \partial\Omega_\Sigma$ also belongs to some $\partial\Omega_j$, $j = 0, \dots, n$

Proposition [Claeys & Hiptmair, 2015]

A tuple in $\mathbb{X}(\Gamma)$ induces unique traces in $\mathbb{H}(\Sigma)$.

The resulting operator $\mathbb{T}: \mathbb{X}(\Gamma) \rightarrow \mathbb{H}(\Sigma)$, $\mathbf{u} \mapsto (\mathbb{T}_D(\mathbf{u}), \mathbb{T}_N(\mathbf{u}))$ is continuous and surjective.

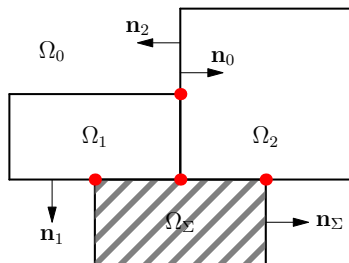
For all $\mathbf{u}, \mathbf{v} \in \mathbb{X}(\Gamma)$ we have $[\mathbf{u}, \mathbf{v}] = -[\mathbb{T}(\mathbf{u}), \mathbb{T}(\mathbf{v})]_\Sigma$

[Claeys & Hiptmair, *Integral Equations for Acoustic Scattering by Partially Impenetrable Composite Objects*, 2015]

Single-Trace FEM-BEM formulation

Goal: stable formulation with **FEM** in Ω_Σ + **BEM** in $\Omega_j, j = 0, \dots, n$ for the transmission problem

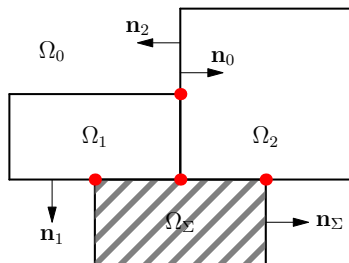
$$\left\{ \begin{array}{l} -\Delta U - \kappa_\Sigma^2(\mathbf{x})U = f \text{ in } \Omega_\Sigma \\ -\Delta U - \kappa_j^2 U = 0 \text{ in } \Omega_j, j = 0, \dots, n \\ U - U_{\text{inc}} \text{ outgoing in } \Omega_0 \\ \gamma_D^j U - \gamma_D^k U = 0 \\ \gamma_N^j U + \gamma_N^k U = 0 \quad \text{on } \partial\Omega_j \cap \partial\Omega_k, j \neq k \\ \gamma_D^j U - \gamma_D^\Sigma U = 0 \\ \gamma_N^j U + \gamma_N^\Sigma U = 0 \quad \text{on } \partial\Omega_j \cap \Sigma \end{array} \right.$$



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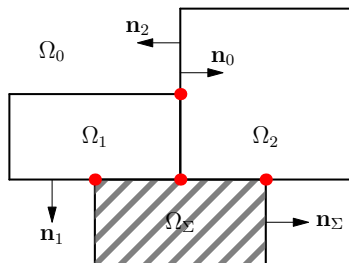
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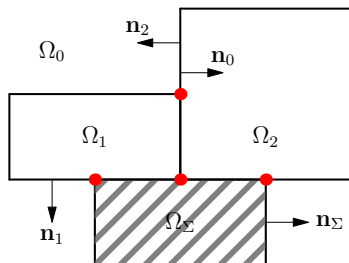
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Single-Trace FEM-BEM formulation

First reformulate the piece-wise homogenous part by BIEs

$$\begin{cases} \gamma_D^j U - \gamma_D^k U = 0 \\ \gamma_N^j U + \gamma_N^k U = 0 \\ \gamma_D^j U - \gamma_D^\Sigma U = 0 \end{cases} \text{ on } \partial\Omega_j \cap \partial\Omega_k, j \neq k$$

Set $\mathbf{u} := (\gamma^0 U, \dots, \gamma^n U)$

Seek

$\mathbf{u} \in \mathbb{X}(\Gamma)$ with $\gamma_D^\Sigma U = T_D(\mathbf{u})$

Single-Trace FEM-BEM formulation

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Seek

$\mathbf{u} \in \mathbb{X}(\Gamma)$ with $\gamma_D^\Sigma U = \mathbb{T}_D(\mathbf{u})$

$$\begin{cases} -\Delta U - \kappa_j^2 U = 0 \text{ in } \Omega_j, j = 0, \dots, n \\ U - U_{\text{inc}} \text{ outgoing in } \Omega_0 \end{cases}$$

$(\mathbf{A} - \text{Id}/2)(\mathbf{u} - \mathbf{u}^{\text{inc}}) = 0$

where $\mathbf{u}^{\text{inc}} := (\gamma^0 U_{\text{inc}}, 0, \dots, 0)$ and $\mathbf{A}: \mathbb{H}(\Gamma) \rightarrow \mathbb{H}(\Gamma)$

$$\mathbf{A}(\mathbf{u}) := (A_{\kappa_j}^j(\mathbf{u}_j))_{j=0}^n = \begin{bmatrix} A_{\kappa_0}^0 & 0 & \dots & 0 \\ 0 & A_{\kappa_1}^1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & A_{\kappa_n}^n \end{bmatrix} \begin{bmatrix} \mathbf{u}_0 \\ \vdots \\ \vdots \\ \mathbf{u}_n \end{bmatrix}$$

Single-Trace FEM-BEM formulation

$\mathbf{u} \in \mathbb{X}(\Gamma)$ with $\gamma_D^\Sigma \mathbf{u} = \mathbf{T}_D(\mathbf{u})$ such that

$$(\mathbf{A} - \text{Id}/2)\mathbf{u} = -\mathbf{u}^{\text{inc}}$$

variational form: $\forall \mathbf{v} \in \mathbb{X}(\Gamma)$ with $\mathbf{T}_D(\mathbf{v}) = \gamma_D^\Sigma \mathbf{v}$

$$\begin{aligned} [\mathbf{A}(\mathbf{u}), \mathbf{v}] - \frac{1}{2}[\mathbf{u}, \mathbf{v}] &= -[\mathbf{u}^{\text{inc}}, \mathbf{v}] \\ &\quad \parallel \\ &\quad + \frac{1}{2}[\mathbf{T}(\mathbf{u}), \mathbf{T}(\mathbf{v})]_\Sigma \end{aligned}$$

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$$[\mathbf{A}(\mathbf{u}), \Theta(\mathbf{v})] + \frac{1}{2}[\mathbb{T}(\mathbf{u}), \mathbb{T}(\Theta(\mathbf{v}))]_\Sigma = -[\mathbf{u}^{\text{inc}}, \Theta(\mathbf{v})]$$

where

$$\Theta_j \begin{pmatrix} v_j \\ q_j \end{pmatrix} := \begin{pmatrix} -v_j \\ q_j \end{pmatrix}, \quad \Theta(\mathbf{v}) := (\Theta_j(\mathbf{v}_j))_{j=0}^n \quad \text{for } \mathbf{v} = \begin{pmatrix} v_j \\ q_j \end{pmatrix}_{j=0}^n \in \mathbb{H}(\Gamma)$$

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Single-Trace FEM-BEM formulation

Now reformulate the heterogeneous part by usual domain var. form.

$$\begin{cases} -\Delta U - \kappa_{\Sigma}^2(\mathbf{x})U = f & \text{in } \Omega_{\Sigma} \\ \gamma_{\mathbf{N}}^j U + \gamma_{\mathbf{N}}^{\Sigma} U = 0 & \text{on } \partial\Omega_j \cap \Sigma, j = 0, \dots, n \end{cases}$$

In Ω_{Σ} : find $U \in H^1(\Omega_{\Sigma})$ s.t.

$$\int_{\Omega_{\Sigma}} (\nabla U \cdot \nabla V - \kappa_{\Sigma}^2(\mathbf{x})UV) \, d\mathbf{x} - \left\langle \gamma_{\mathbf{D}}^{\Sigma} V, \gamma_{\mathbf{N}}^{\Sigma} U \right\rangle_{\Sigma} = \int_{\Omega_{\Sigma}} fV \, d\mathbf{x} \quad \forall V \in H^1(\Omega_{\Sigma})$$

using $\gamma_{\mathbf{N}}^{\Sigma} U = +\mathbf{T}_{\mathbf{N}}(\mathbf{u})$:

$$a_{\Sigma}(U, V) - \left\langle \gamma_{\mathbf{D}}^{\Sigma} V, \mathbf{T}_{\mathbf{N}}(\mathbf{u}) \right\rangle_{\Sigma} = F_{\Sigma}(V) \quad \forall V \in H^1(\Omega_{\Sigma})$$

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using $\gamma_{\mathbf{N}}^{\Sigma} U = +T_{\mathbf{N}}(\mathbf{u})$:

$$a_{\Sigma}(U, V) - \left\langle \gamma_{\mathbf{D}}^{\Sigma} V, T_{\mathbf{N}}(\mathbf{u}) \right\rangle_{\Sigma} = F_{\Sigma}(V) \quad \forall V \in H^1(\Omega_{\Sigma})$$

Single-Trace FEM-BEM formulation

Summing and simplifying

Single-Trace FEM-BEM formulation

find $U \in H^1(\Omega_\Sigma)$, $\mathbf{u} \in \mathbb{X}(\Gamma)$, with $\gamma_D^\Sigma U = T_D(\mathbf{u})$ such that

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Proposition

The solution to the transmission problem is given by

$$\tilde{U}(\mathbf{x}) := U(\mathbf{x}) \quad \text{for } \mathbf{x} \in \Omega_\Sigma,$$

$$\tilde{U}(\mathbf{x}) := (G_{\kappa_0}^0(\mathbf{u}_0) + U_{\text{inc}})(\mathbf{x}) \quad \text{for } \mathbf{x} \in \Omega_0,$$

$$\tilde{U}(\mathbf{x}) := G_{\kappa_j}^j(\mathbf{u}_j)(\mathbf{x}) \quad \text{for } \mathbf{x} \in \Omega_j, j = 1, \dots, n.$$

Single-Trace FEM-BEM formulation

A **generalized Gårding inequality** is satisfied

Consequences: *in the case of injectivity*

- well-posedness
- stability (inf-sup condition)
- for Galerkin, discrete inf-sup condition and quasi-optimal convergence

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Proposition (Injectivity condition)

Let $U \in H^1(\Omega_\Sigma)$, $\mathbf{u} \in \mathbb{X}(\Gamma)$ with $\gamma_D^\Sigma U = T_D(\mathbf{u})$ solve the FEM-BEM STF with $f = 0$, $U_{\text{inc}} = 0$. Then

- $U = 0$
- $\mathbf{u} = 0$ is the unique solution **IFF**

for all $j = 0, \dots, n$, $\Sigma \not\subset \partial\Omega_j$ or $\kappa_j \notin \mathfrak{S}(\Delta, \Omega_\Sigma)$

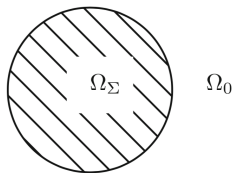
(i.e. κ_j^2 not an interior Dirichlet eigenvalue of $-\Delta$ on Ω_Σ)

Single-Trace FEM-BEM formulation

Spurious resonances examples

Costabel coupling: $\Sigma \subset \partial\Omega_0$

\Rightarrow **spurious resonances** if $\kappa_0 \in \mathfrak{G}(\Delta, \Omega_\Sigma)$

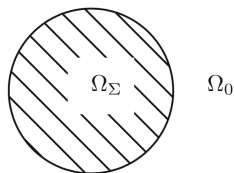


Single-Trace FEM-BEM formulation

Spurious resonances examples

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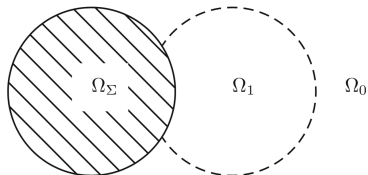
\Rightarrow **spurious resonances** if $\kappa_0 \in \mathfrak{G}(\Delta, \Omega_\Sigma)$



$\mathbb{R}^d = \overline{\Omega}_0 \cup \overline{\Omega}_1 \cup \overline{\Omega}_\Sigma$, but $\kappa_0 = \kappa_1$

$\Sigma \not\subset \partial\Omega_1$ and $\Sigma \not\subset \partial\Omega_0$

\Rightarrow **no spurious resonances** no matter κ_0 !



Results analogue to

[Claeys & Hiptmair, *Integral Equations for Acoustic Scattering by Partially Impenetrable Composite Objects*, 2015]

Outline

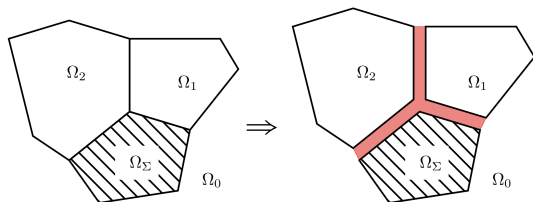
- 1 Introduction
- 2 Recap of boundary integral operators and Costabel coupling
- 3 Single-Trace FEM-BEM formulation
- 4 Multi-Trace FEM-BEM formulation**
- 5 Combined field formulations

Gap setting and new trace spaces

Problem with STF: $\mathbb{X}(\Gamma)$ contains transmission conditions in strong form

- not flexible
- obstacle to operator preconditioning [Claeys, Hiptmair & Jerez-Hanckes, 2013]

⇒ **Multi-Trace Formulations:**



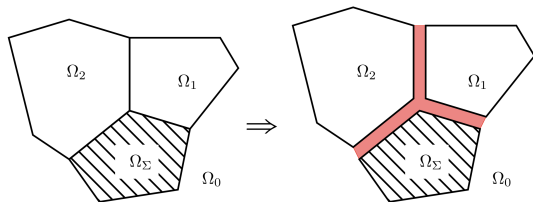
idea: apply STFs to **gap configurations** with vanishing gap

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⇒ **Multi-Trace Formulations:**



idea: apply STFs to **gap configurations** with vanishing gap

$\{(U, u) \in H^1(\Omega_\Sigma) \times \mathbb{X}(\Gamma) \mid \gamma_D^\Sigma U = T_D(u)\}$ isomorphic to $H^1(\Omega_\Sigma) \times \widehat{\mathbb{H}}(\Gamma)$

multi-trace space:

$$\widehat{\mathbb{H}}(\Gamma) = \mathbb{H}(\partial\Omega_1) \times \cdots \times \mathbb{H}(\partial\Omega_n) \times H^{-\frac{1}{2}}(\Sigma)$$

Multi-Trace FEM-BEM formulation

Elaborate the FEM-BEM STF by eliminating all the contributions on $\partial\Omega_0\dots$

Global Multi-Trace FEM-BEM formulation

find $U \in H^1(\Omega_\Sigma)$, $\hat{u} \in \hat{\mathbb{H}}(\Gamma)$, $\hat{u} = (\hat{u}_1, \dots, \hat{u}_n, p_\Sigma)$, such that

$$\begin{aligned} a_\Sigma(U, V) + \{\{\hat{A}(\hat{u}), \Theta(\hat{v})\}\} + \frac{1}{2} \left[\begin{pmatrix} \gamma_D^\Sigma U \\ p_\Sigma \end{pmatrix}, \begin{pmatrix} \gamma_D^\Sigma V \\ q_\Sigma \end{pmatrix} \right]_\Sigma \\ = F_\Sigma(V) + \{\{\hat{f}, \Theta(\hat{v})\}\} \quad \forall V \in H^1(\Omega_\Sigma), \hat{v} \in \hat{\mathbb{H}}(\Gamma), \hat{v} = (\hat{v}_1, \dots, \hat{v}_n, q_\Sigma) \end{aligned}$$

$$\begin{aligned} \text{where } \hat{u} &:= (\hat{u}_1, \dots, \hat{u}_n, (\gamma_D^\Sigma U, p_\Sigma)) \quad \hat{v} := (\hat{v}_1, \dots, \hat{v}_n, (\gamma_D^\Sigma V, q_\Sigma)) \\ \hat{f} &:= (\gamma^1 U_{\text{inc}}, \dots, \gamma^n U_{\text{inc}}, \gamma^\Sigma U_{\text{inc}}) \end{aligned}$$

Skew-symmetric duality pairing: difference with $[\cdot, \cdot]$

$$\{\{u, v\}\} := \sum_{j=1}^n [u_j, v_j]_j + [u_\Sigma, v_\Sigma]_\Sigma$$

Multi-Trace FEM-BEM formulation

$$\hat{\hat{\mathbf{A}}} = \begin{bmatrix} \mathbf{A}_{\kappa_1}^1 + \mathbf{A}_{\kappa_0}^1 & \gamma^1 \mathbf{G}_{\kappa_0}^2 & \dots & \gamma^1 \mathbf{G}_{\kappa_0}^n & \gamma^1 \mathbf{G}_{\kappa_0}^\Sigma \\ \gamma^2 \mathbf{G}_{\kappa_0}^1 & \mathbf{A}_{\kappa_2}^2 + \mathbf{A}_{\kappa_0}^2 & & \gamma^2 \mathbf{G}_{\kappa_0}^n & \gamma^2 \mathbf{G}_{\kappa_0}^\Sigma \\ \vdots & & \ddots & & \vdots \\ \gamma^n \mathbf{G}_{\kappa_0}^1 & \gamma^n \mathbf{G}_{\kappa_0}^2 & & \mathbf{A}_{\kappa_n}^n + \mathbf{A}_{\kappa_0}^n & \gamma^n \mathbf{G}_{\kappa_0}^\Sigma \\ \gamma^\Sigma \mathbf{G}_{\kappa_0}^1 & \gamma^\Sigma \mathbf{G}_{\kappa_0}^2 & \dots & \gamma^\Sigma \mathbf{G}_{\kappa_0}^n & \mathbf{A}_{\kappa_0}^\Sigma \end{bmatrix}$$

global MTF: all subdomains coupled with all other subdomains

Proposition

The solution to the transmission problem is given by

$$\tilde{U}(\mathbf{x}) := U(\mathbf{x}) \quad \text{for } \mathbf{x} \in \Omega_\Sigma$$

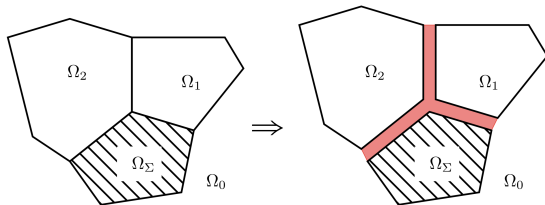
$$\tilde{U}(\mathbf{x}) := \left(U_{\text{inc}} - G_{\kappa_0}^\Sigma \left(\begin{array}{c} \gamma_D^\Sigma U \\ \rho_\Sigma \end{array} \right) - \sum_{j=1}^n G_{\kappa_0}^j(\hat{u}_j) \right)(\mathbf{x}) \quad \text{for } \mathbf{x} \in \Omega_0$$

$$\tilde{U}(\mathbf{x}) := G_{\kappa_j}^j(\hat{u}_j)(\mathbf{x}) \quad \text{for } \mathbf{x} \in \Omega_j, j = 1, \dots, n$$

Proposition

The formulation satisfies a **generalized Gårding inequality**

Multi-Trace FEM-BEM formulation



Gap configuration: $\Sigma \subset \partial\Omega_0 \Rightarrow$ spurious resonances expected!

Proposition (Injectivity condition)

Let $U \in H^1(\Omega_\Sigma)$, $\hat{u} \in \widehat{\mathbb{H}}(\Gamma)$ solve the FEM-BEM MTF with $f = 0$, $U_{\text{inc}} = 0$. Then

- $U = 0$
- $\hat{u} = 0$ is the unique solution **IFF** $\kappa_0 \notin \mathfrak{S}(\Delta, \Omega_\Sigma)$

Outline

- 1 Introduction
- 2 Recap of boundary integral operators and Costabel coupling
- 3 Single-Trace FEM-BEM formulation
- 4 Multi-Trace FEM-BEM formulation
- 5 Combined field formulations**

Combined Field Integral Equations

Goal: remedy for spurious resonances

Combined Field Integral Equations (CFIEs)

Basic idea: Helmholtz BVPs with *Robin boundary conditions* always uniquely solvable (\neq pure Dirichlet problems)

Regularized CFIEs: use *compact operators to map between Dirichlet and Neumann traces* before combining them

For $n = 0$ (Costabel coupling): [Hiptmair & Meury, 2006]

For multi-domain setting: inspiration from [Claeys & Hiptmair, 2015]

Regularizing and trace transformation operators

Regularizing operator: $M: H^{-1/2}(\Sigma) \rightarrow H^{+1/2}(\Sigma)$ that satisfies

- M compact
- $\text{Im}\{\langle M\bar{\varphi}, \varphi \rangle_{\Sigma}\} < 0 \quad \forall \varphi \in H^{-1/2}(\Sigma) \setminus \{0\}$

Example: $M = -\iota(-\Delta_{\Sigma} + \text{Id})^{-1}$, where Δ_{Σ} is the Laplace-Beltrami op.

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Trace spaces

(here $g = \gamma_D^{\Sigma} U$)

$$\mathbb{X}_g(\Gamma) = \{ \mathbf{u} \in \mathbb{X}(\Gamma) \mid T_D(\mathbf{u}) = g \}$$

$$\mathbb{X}_{M,g}(\Gamma) = \{ \mathbf{u} \in \mathbb{X}(\Gamma) \mid T_D(\mathbf{u}) = M T_N(\mathbf{u}) + g \}$$

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Trace transformation operator:

(with E_{Σ} right inverse of γ_D^{Σ})

$$R: \mathbb{X}(\Gamma) \rightarrow \mathbb{X}(\Gamma), \quad R = \text{Id} + C, \quad \text{with } C := \left((\gamma_D^j \circ E_{\Sigma} \circ M \circ T_N, 0) \right)_{j=0}^n$$

Lemma

R isomorphism and $R(\mathbb{X}_g(\Gamma)) = \mathbb{X}_{M,g}(\Gamma)$

Single-Trace Combined Field FEM-BEM formulation

$\mathbf{u} \in \mathbb{X}(\Gamma)$ with $T_D(\mathbf{u}) = \gamma_D^\Sigma U$ such that

$$(A - \text{Id}/2)\mathbf{u} = -\mathbf{u}^{\text{inc}}$$

variational form:

$$[(A - \text{Id}/2)\mathbf{u}, \tilde{\mathbf{v}}] = -[\mathbf{u}^{\text{inc}}, \tilde{\mathbf{v}}] \quad \forall \tilde{\mathbf{v}} \in \mathbb{X}_{M, \bar{\mathbf{g}}}(\Gamma), \quad \bar{\mathbf{g}} = \gamma_D^\Sigma V$$

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$$\tilde{\mathbf{v}} \in \mathbb{X}_{M, \bar{g}}(\Gamma) \iff \tilde{\mathbf{v}} = R\mathbf{v} = (\text{Id} + C)\mathbf{v} \quad \text{for } \mathbf{v} \in \mathbb{X}_{\bar{g}}(\Gamma)$$

$$[(A - \text{Id}/2)\mathbf{u}, \mathbf{v}] + \underbrace{[(A - \text{Id}/2)\mathbf{u}, C\mathbf{v}]}_{=: c(\mathbf{u}, \mathbf{v})} = -[\mathbf{u}^{\text{inc}}, R\mathbf{v}] \quad \forall \mathbf{v} \in \mathbb{X}_{\bar{g}}(\Gamma)$$

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$$[A(\mathbf{u}), \mathbf{v}] + \frac{1}{2}[T(\mathbf{u}), T(\mathbf{v})]_\Sigma + c(\mathbf{u}, \mathbf{v}) = -[\mathbf{u}^{\text{inc}}, R\mathbf{v}] \quad \forall \mathbf{v} \in \mathbb{X}_{\bar{g}}(\Gamma)$$

Single-Trace Combined Field FEM-BEM formulation

Single-Trace Combined Field FEM-BEM formulation

find $U \in H^1(\Omega_\Sigma)$, $u \in \mathbb{X}(\Gamma)$, with $\gamma_D^\Sigma U = T_D(u)$ such that

$$a_\Sigma(U, V) + [A(u), \Theta(v)] + c(u, \Theta(v)) + \frac{1}{2} \left[\begin{pmatrix} \gamma_D^\Sigma U \\ T_N(u) \end{pmatrix}, \begin{pmatrix} \gamma_D^\Sigma V \\ T_N(v) \end{pmatrix} \right]_\Sigma \\ = F_\Sigma(V) - [u^{\text{inc}}, R(\Theta(v))]$$

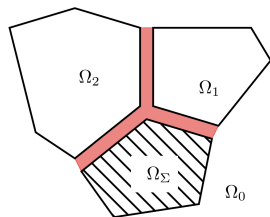
$\forall V \in H^1(\Omega_\Sigma)$, $v \in \mathbb{X}(\Gamma)$ with $\gamma_D^\Sigma V = T_D(v)$

compact perturbation of previous FEM-BEM STF!

\Rightarrow generalized Gårding inequality

Multi-Trace Combined Field FEM-BEM formulation

Manipulate the STF formulation *in the gap setting*

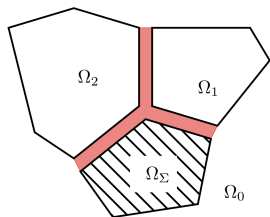


Look for \widehat{c} such that $\widehat{c}((U, \widehat{u}), (V, \widehat{v})) = c(\mathbf{u}, \mathbf{v})$ where

$$(U, \widehat{u}), (V, \widehat{v}) \in H^1(\Omega_\Sigma) \times \widehat{\mathbb{H}}(\Gamma) \longleftrightarrow \mathbf{u}, \mathbf{v} \in \mathbb{X}(\Gamma) \text{ with } \gamma_D^\Sigma U = T_D(\mathbf{u})$$

Multi-Trace Combined Field FEM-BEM formulation

Manipulate the STF formulation *in the gap setting*



Extension operator E_Σ can be picked s.t.

$$\gamma_D^j \circ E_\Sigma = 0 \text{ for } j \neq 0$$

$$C = \left((\gamma_D^j \circ E_\Sigma \circ M \circ T_N, 0) \right)_{j=0}^n$$

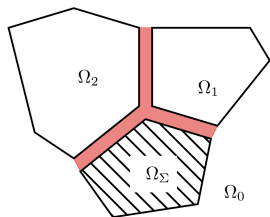
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Look for \widehat{c} such that $\widehat{c}((U, \hat{u}), (V, \hat{v})) = c(u, v)$ where

$$(U, \hat{u}), (V, \hat{v}) \in H^1(\Omega_\Sigma) \times \widehat{\mathbb{H}}(\Gamma) \longleftrightarrow u, v \in \mathbb{X}(\Gamma) \text{ with } \gamma_D^\Sigma U = T_D(u)$$

$$c(u, v) = \sum_{j=0}^n \left[\gamma_c^j G_{\kappa_j}^j(u_j), (Cv)_j \right]_j = [\gamma_c^0 G_{\kappa_0}^0(u_0), (Cv)_0]_0 = \langle \gamma_N^\Sigma G_{\kappa_0}^0(u_0), MT_N(v) \rangle_\Sigma$$

Multi-Trace Combined Field FEM-BEM formulation

$$c(\mathbf{u}, \mathbf{v}) = \langle \mathbf{M}^* \gamma_N^\Sigma G_{\kappa_0}^0(\mathbf{u}_0), \mathbf{T}_N(\mathbf{v}) \rangle_\Sigma$$

$\mathbf{u} \in \mathbb{X}(\Gamma)$ with $\mathbf{T}_D(\mathbf{u}) = \gamma_D^\Sigma U$ so

$$G_{\kappa_0}^0(\mathbf{u}_0) = -G_{\kappa_0}^\Sigma \left(\begin{array}{c} \gamma_D^\Sigma U \\ \mathbf{T}_N(\mathbf{u}) \end{array} \right) - \sum_{j=1}^n G_{\kappa_0}^j(\mathbf{u}_j)$$

Multi-Trace Combined Field FEM-BEM formulation

$$c(\mathbf{u}, \mathbf{v}) = \langle \mathbf{M}^* \gamma_N^\Sigma G_{\kappa_0}^0(\mathbf{u}_0), \mathbf{T}_N(\mathbf{v}) \rangle_\Sigma$$

$\mathbf{u} \in \mathbb{X}(\Gamma)$ with $\mathbf{T}_D(\mathbf{u}) = \gamma_D^\Sigma U$ so

$$G_{\kappa_0}^0(\mathbf{u}_0) = -G_{\kappa_0}^\Sigma \left(\begin{array}{c} \gamma_D^\Sigma U \\ \mathbf{T}_N(\mathbf{u}) \end{array} \right) - \sum_{j=1}^n G_{\kappa_0}^j(\mathbf{u}_j)$$

Therefore

$$\widehat{c}((U, \hat{\mathbf{u}}), (V, \hat{\mathbf{v}})) = - \left\langle \mathbf{M}^* \gamma_N^\Sigma G_{\kappa_0}^\Sigma \left(\begin{array}{c} \gamma_D^\Sigma U \\ \rho_\Sigma \end{array} \right), q_\Sigma \right\rangle_\Sigma - \sum_{j=1}^n \left\langle \mathbf{M}^* \gamma_N^\Sigma G_{\kappa_0}^j(\hat{\mathbf{u}}_j), q_\Sigma \right\rangle_\Sigma$$

Multi-Trace Combined Field FEM-BEM formulation

$$\widehat{\mathbb{C}}((U, \hat{u}), (V, \hat{v})) = - \left\langle \mathbf{M}^* \gamma_{\mathbb{N}}^{\Sigma} \mathbf{G}_{\kappa_0}^{\Sigma} \begin{pmatrix} \gamma_{\mathbb{D}}^{\Sigma} U \\ \rho_{\Sigma} \end{pmatrix}, q_{\Sigma} \right\rangle_{\Sigma} - \sum_{j=1}^n \left\langle \mathbf{M}^* \gamma_{\mathbb{N}}^{\Sigma} \mathbf{G}_{\kappa_0}^j(\hat{u}_j), q_{\Sigma} \right\rangle_{\Sigma}$$

Therefore

$$\widehat{\mathbf{A}}_{\mathbf{M}} = \begin{bmatrix} \mathbf{A}_{\kappa_1}^1 + \mathbf{A}_{\kappa_0}^1 & \gamma^1 \mathbf{G}_{\kappa_0}^2 & \dots & \gamma^1 \mathbf{G}_{\kappa_0}^n & \gamma^1 \mathbf{G}_{\kappa_0}^{\Sigma} \\ \gamma^2 \mathbf{G}_{\kappa_0}^1 & \mathbf{A}_{\kappa_2}^2 + \mathbf{A}_{\kappa_0}^2 & & \gamma^2 \mathbf{G}_{\kappa_0}^n & \gamma^2 \mathbf{G}_{\kappa_0}^{\Sigma} \\ \vdots & & \ddots & & \vdots \\ \gamma^n \mathbf{G}_{\kappa_0}^1 & \gamma^n \mathbf{G}_{\kappa_0}^2 & & \mathbf{A}_{\kappa_n}^n + \mathbf{A}_{\kappa_0}^n & \gamma^n \mathbf{G}_{\kappa_0}^{\Sigma} \\ \left(\gamma_{\mathbb{D}}^{\Sigma} - \mathbf{M}^* \gamma_{\mathbb{N}}^{\Sigma} \right) \mathbf{G}_{\kappa_0}^1 & \left(\gamma_{\mathbb{D}}^{\Sigma} - \mathbf{M}^* \gamma_{\mathbb{N}}^{\Sigma} \right) \mathbf{G}_{\kappa_0}^2 & \dots & \left(\gamma_{\mathbb{D}}^{\Sigma} - \mathbf{M}^* \gamma_{\mathbb{N}}^{\Sigma} \right) \mathbf{G}_{\kappa_0}^n & \mathbf{A}_{\kappa_0}^{\Sigma} - \begin{pmatrix} \mathbf{M}^* \gamma_{\mathbb{N}}^{\Sigma} \\ 0 \end{pmatrix} \mathbf{G}_{\kappa_0}^{\Sigma} \end{bmatrix}$$

Multi-Trace Combined Field FEM-BEM formulation

Global Multi-Trace Combined Field FEM-BEM formulation

find $U \in H^1(\Omega_\Sigma)$, $\hat{u} \in \hat{\mathbb{H}}(\Gamma)$, $\hat{u} = (\hat{u}_1, \dots, \hat{u}_n, p_\Sigma)$, such that

$$a_\Sigma(U, V) + \{\{\hat{\mathbb{A}}_M(\hat{u}), \Theta(\hat{v})\}\} + \frac{1}{2} \left[\begin{pmatrix} \gamma_D^\Sigma U \\ p_\Sigma \end{pmatrix}, \begin{pmatrix} \gamma_D^\Sigma V \\ q_\Sigma \end{pmatrix} \right]_\Sigma \\ = F_\Sigma(V) + \{\{\hat{f}_M, \Theta(\hat{v})\}\} \quad \forall V \in H^1(\Omega_\Sigma), \hat{v} \in \hat{\mathbb{H}}(\Gamma), \hat{v} = (\hat{v}_1, \dots, \hat{v}_n, q_\Sigma)$$

where $\hat{u} := (\hat{u}_1, \dots, \hat{u}_n, (\gamma_D^\Sigma U, p_\Sigma))$ $\hat{v} := (\hat{v}_1, \dots, \hat{v}_n, (\gamma_D^\Sigma V, q_\Sigma))$

$$\hat{f}_M := \left(\gamma^1 U_{\text{inc}}, \dots, \gamma^n U_{\text{inc}}, \left(\gamma_D^{\Sigma - M^*} \gamma_N^\Sigma \right) U_{\text{inc}} \right)$$

$\hat{\mathbb{A}}_M$ compact perturbation of $\hat{\mathbb{A}} \Rightarrow$ generalized Gårding inequality

Proposition

A solution to the *combined field* formulation is also a solution to the standard formulation

⇒ formulation ok for *general geometric settings*

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⇒ formulation ok for *general geometric settings*

Proposition (Injectivity)

Let $U \in H^1(\Omega_\Sigma)$, $\hat{u} \in \widehat{\mathbb{H}}(\Gamma)$ solve the combined field FEM-BEM MTF formulation with $f = 0$, $U_{\text{inc}} = 0$.

Then $U = 0$, $\hat{u} = 0$.

⇒ all good consequences of Gårding + injectivity!

Conclusion and outlook

- Stable FEM-BEM formulations for multi-domain acoustic scattering, with junction points
- Generalized Gårding inequalities, injectivity conditions
- Combined field versions immune to spurious resonances

Future work:

- piece-wise constant coefficient in Neumann transmission conditions
- local FEM-BEM MTFs \rightarrow optimized Schwarz methods
- quasi-local FEM-BEM MTFs
- other FEM-BEM coupling strategies
- FEM-BEM preconditioners